

Holography and renormalization in Lorentzian signature

Albion Lawrence and Amit Sever

*Brandeis Theory Group, Martin Fisher School of Physics, Brandeis University
Waltham, MA 02454-9110, U.S.A.
E-mail: albion@brandeis.edu, asever@brandeis.edu*

ABSTRACT: De Boer et. al. have found an asymptotic equivalence between the Hamilton-Jacobi equations for supergravity in $(d+1)$ -dimensional asymptotic anti-de Sitter space, and the Callan-Symanzik equations for the dual d -dimensional perturbed conformal field theory. We discuss this correspondence in Lorentzian signature. We construct a gravitational dual of the generating function of correlation functions between initial and final states, in accordance with the construction of Marolf, and find a class of states for which the result has a classical supergravity limit. We show how the data specifying the full set of solutions to the second-order supergravity equations of motion are described in the field theory, despite the first-order nature of the renormalization group equations for the running couplings: one must specify both the couplings and the states, and the latter affects the solutions to the Callan-Symanzik equations.

KEYWORDS: AdS-CFT Correspondence, Renormalization Group.

JHEP10(2006)013

Contents

1. Introduction	2
2. Review of Hamilton-Jacobi theory	4
2.1 Example: the upside-down harmonic oscillator	5
3. AdS/CFT in Lorentzian signature	6
3.1 AdS/CFT basics	7
3.2 Boundary behavior of classical fields in AdS	9
3.3 Quantum states in the CFT	10
3.4 SUGRA field eigenstates and gravitational backreaction	12
3.5 Perturbing the CFT	13
3.5.1 The limit of linearized supergravity	13
3.5.2 Perturbations in the interacting theory	14
3.6 The generating function of correlation functions	14
4. Holographic renormalization and the Hamilton-Jacobi equation	15
4.1 General discussion	15
4.2 The radial Hamilton-Jacobi equation	16
4.3 Solving the Hamilton-Jacobi equations	17
4.4 Relation to the renormalization group	19
4.4.1 Beta functions	19
4.4.2 The Callan-Symanzik equation	21
4.4.3 Caveats	23
5. Holographic RG in Lorentzian signature	23
5.1 The Hamilton-Jacobi functional in Lorentzian signature	24
5.2 The Callan-Symanzik equation for nontrivial matrix elements	25
5.3 Lorentzian HJ/CS correspondence	27
5.4 Perturbed CFTs with an IR cutoff	30
5.5 Extension of the holographic RG equations into the IR	33
5.6 Reversibility of holographic RG	35
6. Conclusions	35

1. Introduction

In the $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence [1], the coordinate position of an excitation relative to the timelike boundary of AdS is in some sense dual to the characteristic scale size of that excitation in the d -dimensional CFT. This can be seen from entropic considerations [2], from the duals of classical bulk probes [3–7]-[8], and from the semiclassical bulk description of Wilson lines [9, 10].

Let \mathbf{x} be coordinates parallel to the boundary and r the coordinate running perpendicular to the boundary. In the Euclidean version of the correspondence, the bulk fields $\phi^a(\mathbf{x}, r)$ are taken to be dual to the (in general space-time dependent) couplings $\lambda^a(\mathbf{x})$ of the boundary theory.¹ The equations of motion for ϕ can be written as an equation for evolution in r , where the boundary of AdS space is dual to the UV of the field theory. The "holographic renormalization group" [11–15]-[16] then relates this evolution to the running of the dual couplings under change of renormalization group scale.

In particular, de Boer, Verlinde, and Verlinde have shown that as we approach the AdS boundary, the Hamilton-Jacobi (HJ) equations for radial evolution of the bulk supergravity fields are equivalent to Callan-Symanzik (CS) equations for the correlation functions of the boundary field theory [15, 16]. In this formalism, all nonsingular solutions to the supergravity equations (such as domain walls in AdS), and solutions with singularities that are resolved by string theory, are manifestly dual to renormalization group flows.

Nonetheless, this and other versions of holographic RG raise a number of issues, of which we list three:

1. The CS equations are first order in the RG scale parameter, while the spacetime equations of motion are second order. In Lorentzian signature, the spacetime equations of motion seem to require twice as many initial conditions [7, 17] than are typically specified in renormalization group flows.
2. As we approach the AdS boundary, $\phi(r)$ in general becomes the coupling of the dual operator at scale $\ell(r)$ (for low-dimension operators this does not have to be true [7, 18].) This is not obviously the right map deep in the interior of AdS spacetimes [19, 20]. Furthermore, in Lorentzian signature [7, 17], ϕ is determined by both the coupling and the state of the dual field theory.
3. Considered as flow equations in the radial direction r of AdS, the spacetime equations of motion are reversible. The Callan-Symanzik equations are also reversible in RG scale. On the other hand, the Wilsonian version of the renormalization group is an evolution under coarse graining, which is not reversible. How then does the Wilsonian picture fit into the AdS/CFT correspondence?

The main goal of this paper is to solve the puzzle raised in the first point. The summary of the solution is as follows. In Lorentzian signature, the second order supergravity equations of motion have two classes of nonsingular solutions characterized by their behavior at the timelike AdS boundary [17, 7]. One class is dual to deformations of the

¹Here \mathbf{x} parameterizes a point on the d -dimensional boundary.

field theory Lagrangian. The second class depends on the semiclassical excitations of the field theory above the vacuum state.² A general solution to the spacetime equations of motion will have terms with both types of boundary behavior, and so be specified both by the couplings of the perturbed CFT, and by the state of the CFT, when the state is a "classical" state in the large-N limit.

In the Hamilton-Jacobi formulation, the equations of motion are solved by first solving the Hamilton-Jacobi equations for Hamilton's principal function S :

$$H(\phi^a, \pi_{a,\phi} = \frac{\partial S}{\partial \phi^a}) = 0$$

and then solving Hamilton's equations

$$\dot{\phi}^a = \frac{\partial H}{\partial \pi_{a,\phi}}(\pi_a = \frac{\partial S}{\partial \phi^a}),$$

where ϕ^a are some fields in AdS, $\pi_{a,\phi}$ are the conjugate field momenta and the dot stands for radial derivative. In holographic renormalization the former equations are dual to the Callan-Symanzik equations, while the latter are dual to the RG equations

$$\Lambda \partial_\Lambda \lambda^a = \beta^a(\lambda)$$

for the couplings λ , where β^a are the beta functions and Λ is a momentum space cutoff. We will show that in these solutions, the freedom to excite modes dual to a choice of state is captured in the choice of solutions to the Hamilton-Jacobi equation. For the holographic equivalence between the Hamilton-Jacobi and Callan-Symanzik equations to hold, we must be able to solve the latter for any choice of state. We find that we can, once we take into account the modification of the Callan-Symanzik equation for correlation functions in a nonvacuum state.

Along the way we will also discuss the problem raised in (2), and resolve the problem raised in (3).

The format of this paper is as follows. In section 2 we review HJ theory and apply it to some simple examples, pointing out specific features useful for our discussion. In section 3 following [7, 17, 21], we discuss the AdS/CFT correspondence in Lorentzian signature. In section 3.1 we review the AdS/CFT basics. In section 3.2 the boundary behavior of classical fields in AdS is discussed. In section 3.3 we explain the classical supergravity manifestation of the CFT states. We find that in order to guarantee the existence of a saddle point over a range of couplings, the eigenstates of the field operators are good choices for initial and final states. In section 3.4 we discuss the issue of gravitational backreaction for such states. In section 3.5 we review the correspondence where the CFT is perturbed by relevant operators. Finally in section 3.6 we discuss the generating function of correlation functions. section 4 is a review and critical discussion of the formalism of [15, 16]. In section 5 we develop a Lorentzian-signature version of the picture in [15, 16]. We

²In this class there are also singular solutions in either signature generated by sources such as D-branes or D-instantons.

identify the "missing constants of motion" in the RG equations with the choice of classical state of the system. In section 5.1 we discuss the Hamilton-Jacobi equations in Lorentzian signature. In section 5.2 we derive a Callan-Symanzik equation for general matrix elements of time-ordered products of operators. In section 5.3 we show that the Hamilton-Jacobi and Callan-Symanzik equations are determined by the same information, thus solving the puzzle posed in question (1). In section 5.4 we discuss an alternate solution to the Hamilton-Jacobi equations, in which the constants of motion are the operator expectation values specified at an infrared cutoff. In section 5.5 we discuss the extension of our story deep into the infrared. In section 5.6 we discuss the degree to which holographic RG is related to the Wilsonian picture of renormalization. In section 6 we conclude.

2. Review of Hamilton-Jacobi theory

Consider a dynamical system with $2n$ phase space variables (\mathbf{q}, \mathbf{p}) , corresponding to positions $\mathbf{q} = \{q_i, i = 1 \dots n\}$ and canonical conjugate momenta $\mathbf{p} = \{p_i, i = 1 \dots n\}$, and a Hamiltonian $H(\mathbf{p}, \mathbf{q})$. In Hamilton-Jacobi theory, the equations of motion are solved in two stages. First, one solves the Hamilton-Jacobi equation for Hamilton's principal function S :

$$\partial_t S(\mathbf{q}, t) + H\left(\mathbf{p} = \frac{\partial S(\mathbf{q}, t)}{\partial \mathbf{q}}, \mathbf{q}\right) = 0. \tag{2.1}$$

This is a nonlinear equation; in general it has many solutions. Given a solution, one finds the classical trajectories $\mathbf{q}(t)$ from a set of first-order differential equations:

$$\dot{\mathbf{q}} = \left. \frac{\partial H}{\partial \mathbf{p}} \right|_{\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}}. \tag{2.2}$$

If H is quadratic in momenta, the full equations of motion for q are second order in time. Their full solutions require that one specify $2n$ constants of motion (\mathbf{a}, \mathbf{b}) , where $\mathbf{a} = \{a_i, i = 1 \dots n\}$ and $\mathbf{b} = \{b_i, i = 1 \dots n\}$. For example, $\mathbf{q}(t_i)$ and $\dot{\mathbf{q}}(t_i)$ at some initial time t_i determine the trajectory completely. On the other hand, a full solution to (2.2) requires only n constants of motion (\mathbf{b}) , for example $\mathbf{b} = \mathbf{q}(t_i)$ at some initial time t_i .

The point is that the additional constants of motion of the dynamics are contained in the choice of solution to (2.1). The solution can be written as $S(\mathbf{q}(t), \mathbf{a}, t)$, where \mathbf{a} are n constants of motion. If in addition to (2.2) we demand that

$$\left| \frac{\partial^2 S(\mathbf{q}(t), \mathbf{a}; t)}{\partial q^i \partial a^j} \right| \neq 0, \tag{2.3}$$

then the constants of motion \mathbf{b} are given by

$$\mathbf{b} = - \frac{\partial S(\mathbf{q}(t), \mathbf{a}, t)}{\partial \mathbf{a}}, \tag{2.4}$$

\mathbf{b} is canonically conjugate to \mathbf{a} and S is the generating function of canonical transformation between (\mathbf{q}, \mathbf{p}) and (\mathbf{a}, \mathbf{b}) . Since the new canonical variables (\mathbf{a}, \mathbf{b}) are constants of motion, the new Hamiltonian

$$K(\mathbf{a}, \mathbf{b}, t) = \partial_t S + H = 0. \tag{2.5}$$

Note that, instead of using (2.2) one can extract the solution $\mathbf{q}(\mathbf{a}, \mathbf{b}, t)$ directly from (2.4).

One particular choice of \mathbf{a} is $\mathbf{q}(t_0)$ at some initial time t_0 . The corresponding solution to (2.1) is:

$$S(\mathbf{q}(t), \mathbf{q}(t_0), t) = \int_{t_0}^t dt' (\mathbf{p}\dot{\mathbf{q}} - H), \quad (2.6)$$

evaluated on a solution to the classical equations of motion with fixed $\mathbf{q}(t_0) \equiv \mathbf{q}_0 = \mathbf{a}$.

2.1 Example: the upside-down harmonic oscillator

As an example, let us study the one-dimensional upside-down harmonic oscillator, with Hamiltonian $H = \frac{1}{2}p^2 - \frac{1}{2}\Omega^2q^2$. Since H is time independent, one solution to (2.1) can be found by setting $S_1 = -Et + W(q)$. The HJ equation becomes a differential equation for W :

$$\frac{1}{2} [(\partial_q W)^2 - \Omega^2 q^2] = E, \quad (2.7)$$

which has the solution:

$$W(q, E) = \frac{E}{\Omega} \sinh^{-1} \left(\frac{\Omega q}{\sqrt{2E}} \right) + \frac{\Omega}{2} q \sqrt{\frac{2E}{\Omega^2} + q^2} + f_1(E). \quad (2.8)$$

Here $\mathbf{a} = E$ is the constant of motion governing the solution to the Hamilton-Jacobi equation. $f_1(E)$ is an arbitrary function; f_1 changes the definition of the phase space variable conjugate to E . The equation of motion for q now reduces to:

$$\dot{q} = p = \partial_q W = \sqrt{2E + \Omega^2 q^2}, \quad (2.9)$$

and has the solution

$$q(t) = \frac{\sqrt{2E}}{\Omega} \sinh [\Omega(t - t_0)]. \quad (2.10)$$

Here

$$\mathbf{b} \equiv -\frac{\partial S_1}{\partial E} = t_0 - f_1'(E) \quad (2.11)$$

is the integration constant arising from the first order differential equation (2.9). The complete solution is specified by t_0 and E , where t_0 is defined as the time at which $q(t_0) = 0$. The solution S_1 is the generating function of the canonical transformation between (q, p) and $(E, t_0 - f_1'(E))$.

Alternatively, we can find the classical action for $q(t)$ given that $q(t_0) = q_0$. Here $\mathbf{a} = q_0$ is the constant of motion that arises in the solution to (2.1):

$$S_2(q, q_0, t) = \frac{1}{2}\Omega \coth [\Omega(t - t_0)] (q^2 + q_0^2) - \Omega \operatorname{csch} [\Omega(t - t_0)] qq_0 + f_2(q_0). \quad (2.12)$$

where f_2 is an arbitrary function, changing the definition of the momentum conjugate to q_0 . Eq. (2.12) can be computed by simply inserting the known classical solution into the classical action $S = \int_{t_0}^t L$. The equation of motion for q is:

$$\dot{q} = \partial_q S_2 = \Omega \coth [\Omega(t - t_0)] q - \Omega \operatorname{csch} [\Omega(t - t_0)] q_0. \quad (2.13)$$

Integrating this, we find that

$$q(t) = q_0 \cosh [\Omega(t - t_0)] + \frac{p_0}{\Omega} \sinh [\Omega(t - t_0)] . \quad (2.14)$$

Here

$$\mathbf{b} \equiv -\frac{\partial S_2}{\partial q_0} = p_0 - f'_2(q_0) \quad (2.15)$$

is the integration constant arising from the first order equation (2.13). S_2 is the generating function of the canonical transformation between (q, p) and $(q_0, p_0 - f'_2(q_0))$.

We have solved for the dynamics by first choosing the constants of motion \mathbf{a} in the solution to the Hamilton-Jacobi equation, and then solving for the trajectory $q(t)$ via Hamilton's equations. Note, however, that for a given \mathbf{a} , not all values of q may be allowed. For example, consider trajectories with fixed energy for the standard harmonic oscillator with frequency ω . The solution can be found from (2.8) by setting $\Omega = i\omega$. For fixed ω , the region $q > 2E/(m\omega^2)$ is classically forbidden. This appears already at the level of the solution W , which becomes imaginary in this region. While W in this region can be used in a WKB analysis (as the phase of the WKB wavefunction satisfies the Hamilton-Jacobi equation to lowest order in \hbar), it does not correspond to any classical trajectory.

3. AdS/CFT in Lorentzian signature

We will be discussing the gravitational duals of perturbed d -dimensional conformal field theories. Consider a CFT \mathcal{X} perturbed by spacetime dependent couplings $\lambda^a(x)$ to local operators $\mathcal{O}_a(x)$. Correlation functions of local operators can be extracted from the transition amplitudes of the perturbed theory:

$$Z[\{\lambda(x)\}] = \langle \psi_+(t_+) | T \exp \left(\frac{-i}{\hbar} \int_{t=t_-}^{t=t_+} d^d x \sum_a \lambda^a(x) \mathcal{O}_a(x) \right) | \psi_-(t_-) \rangle , \quad (3.1)$$

by taking functional derivatives with respect to $\lambda^a(x)$. Here the local operators $\mathcal{O}_a(x)$ and the states $|\psi_{\pm}\rangle$ are written in the interaction picture.

The generating function of Euclidean correlators was constructed in [22, 23]. This study of the duality in Lorentzian signature was initiated in [7, 17], which provided a duality map for the propagating classical and quantum states. In [21] these states were constructed in the bulk, so that they are independent of variations of λ in the interior of $[t_-, t_+]$. This allows (3.1) to be the generating function of matrix elements of time-ordered products of operators.

This section will be dedicated to sketching the gravitational dual of (3.1) in the semi-classical limit, following [7, 17, 21]. Following [21], we take care to define $|\psi_{\pm}\rangle$ so that they are independent of the coupling. In particular, in later sections we will be interested in applying this formula in a classical limit where Z can be considered as a solution to the Hamilton-Jacobi equation of the dual supergravity theory, with $\lambda^a(x)$ as the configuration space variables. We can apply classical Hamilton-Jacobi theory to (3.1) if there is a solution to the classical supergravity equations for each value of $\lambda(x)$. We will argue that for

these purposes, choosing $|\psi_{\pm}\rangle$ to be eigenstates of the field operators in the gravitational dual will lead to Z having the desired properties, and are a technically convenient choice. These states are potentially dangerous in a theory of quantum gravity. We will discuss these dangers and the reasons why they should not trouble the semiclassical computation of (3.1).

We will take the CFT to be $d = 4$, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $U(N)$. The story for other CFTs will be essentially the same, even in other spacetime dimensions.

3.1 AdS/CFT basics

We begin by reviewing general aspects of this duality, as outlined by [1, 22, 23]. The correspondence can be considered in various coordinate patches of AdS spacetime. A set of coordinates which cover the entire Lorentzian spacetime are:

$$ds^2 = - \left(1 + \frac{r^2}{R_{\text{AdS}}^2} \right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{R_{\text{AdS}}^2} \right)} + r^2 d\Omega_3^2, \quad (3.2)$$

where $d\Omega_3$ is the solid angle on S^3 , and R_{AdS} is the AdS radius of curvature. There is a timelike boundary at $r = \infty$ which is conformally equivalent to $\mathbb{R}_t \times S^3$. This boundary is at infinite proper distance along r , but light rays can reach the boundary in finite global time (see figure 1.) String theory on the full $AdS_5 \times S^5$ is dual to $d = 4$, $\mathcal{N} = 4$ super-Yang-Mills theory on $\mathbb{R}_t \times S^3$.

Similarly, we can consider the correspondence for the Poincaré patch of AdS space, described by the metric

$$ds^2 = R_{\text{AdS}}^2 \frac{dz^2 + d\mathbf{x}^2}{z^2}, \quad (3.3)$$

where $d\mathbf{x}^2$ is the metric on four-dimensional Minkowski space $\mathbb{R}^{3,1}$. In these coordinates the timelike boundary is at $z = 0$ (see Fig 2.) String theory on this space is dual to the above 4d CFT on $\mathbb{R}^{3,1}$.

The natural scales in these compactifications are the string scale ℓ_s , the five-dimensional Planck scale ℓ_p , and the radius of curvature of the spacetime R_{AdS} . From these we can form two independent dimensionless ratios, which are dual to dimensionless parameters in the Yang-Mills theory: $(R_{\text{AdS}}/\ell_p)^3 = N^2$ and $(R_{\text{AdS}}/\ell_s)^4 = \lambda \equiv g_{\text{YM}}^2 N$, where g_{YM}^2 is the dimensionless coupling of the gauge theory.

In the large N limit, at fixed Yang-Mills coupling g_{YM} , the low energy supergravity limit of string theory on $AdS_5 \times S^5$ is a good approximation. In this situation, local, low-dimension single-trace operators \mathcal{O}_a are dual to supergravity fields ϕ^a . Among these operators are the 4d stress tensor, which is dual to the 5d metric in an appropriate gauge. We will focus on perturbations by these operators, in particular operators dual to the 5d metric and 5d scalar fields. Single-trace local operators with dimension of order $\lambda^{1/4}$ are dual to massive string states. Deformations by local multi-trace operators [24–26]-[27] can be described by a particular deformation of the boundary conditions. The dual description of Wilson line operators have also been constructed [9, 10].

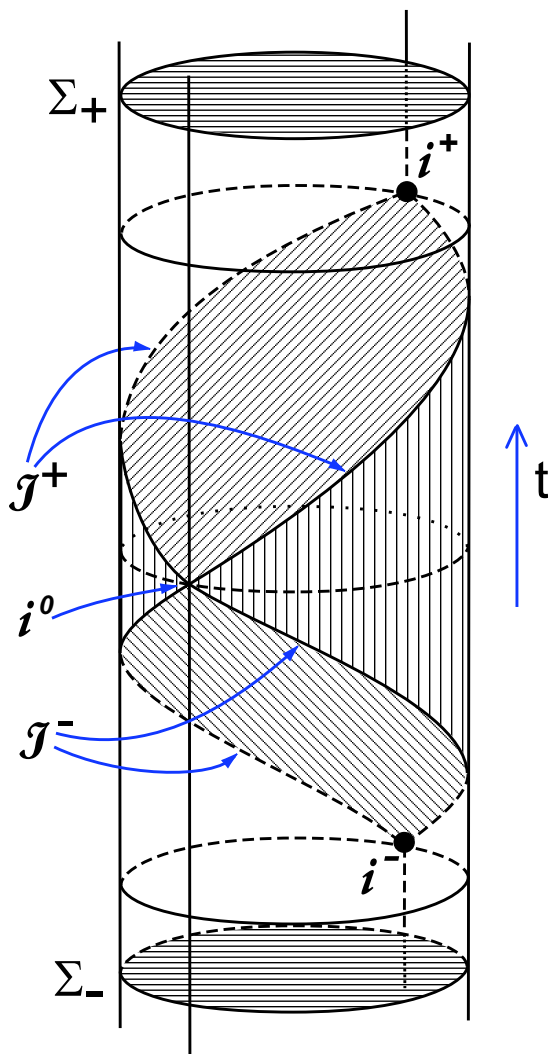


Figure 1: A Penrose diagram for AdS_{d+1} . The cylindrical boundary conformal to $S^{d-1} \times \mathbb{R}$. t is the time in global coordinates. Σ_{\pm} are spacelike slices at times $t_+ > t_-$ (perhaps with $t_{\pm} \rightarrow \pm\infty$): one may define initial and final states of quantum fields in the Heisenberg picture by writing a wavefunction of field values on Σ_- and Σ_+ , respectively. The patch of AdS covered by Poincaré coordinates is the shaded region bounded by $\mathcal{I}^+ \cup \mathcal{I}^- \cup \mathbf{i}^+ \cup \mathbf{i}^- \cup \mathbf{i}^0$.

In standard treatments, which we will follow here, scalar fields in the $5d$ gravitational theory are taken to be dimensionless, and the bulk effective action scales as $N^2 = (R_{\text{AdS}}/\ell_p)^3$. The mass m^2 of these scalars is related to the conformal dimension $\Delta_{a,+}$ of the dual operators by [22, 23]:

$$\Delta_{a,\pm} = 2 \pm \sqrt{4 + R_{\text{AdS}}^2 m^2} . \tag{3.4}$$

The bound $R_{\text{AdS}}^2 m^2 \geq -4$, required for the operator in the CFT to have real conformal dimension, coincides with the lower bound on scalar masses required for stability of the bulk theory [28, 29].

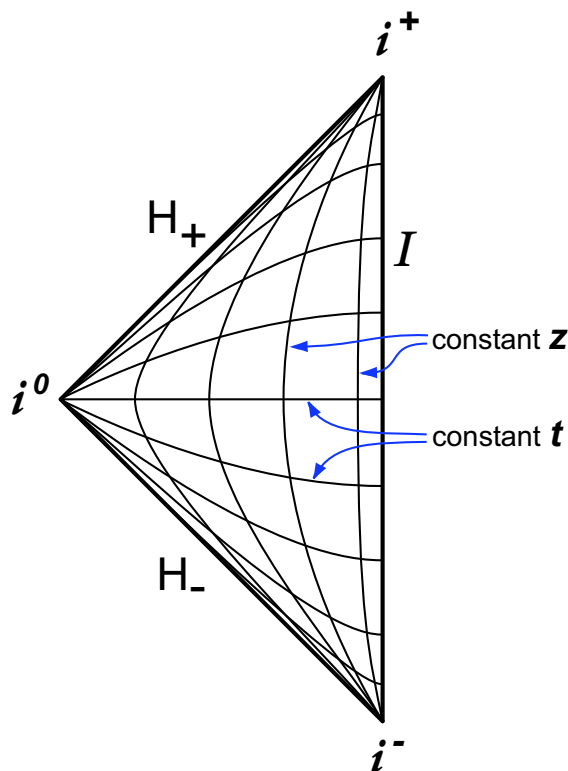


Figure 2: The Penrose diagram for AdS_d in Poincaré coordinates. The boundary \mathcal{I} is the timelike boundary of AdS, and is conformal to $\mathbb{R}^{d-1,1}$. H_{\pm} are coordinate horizons. One may specify the quantum states as $t \rightarrow \pm\infty$ in the bulk with data on $\Sigma_{\pm} = i_{\pm} \cup H_{\pm}$.

3.2 Boundary behavior of classical fields in AdS

Consider small fluctuations of classical supergravity scalar fields, which are well-described by linearized classical supergravity as $N, \lambda \rightarrow \infty$. In Lorentzian signature, at fixed AdS momentum (along $S^3 \times \mathbb{R}_t$ in (3.2), or along $\mathbb{R}_x^{1,3}$ in (3.3)) there are two independent solutions to the linearized equations of motion, which are classified by their boundary behavior as $z \rightarrow 0/r \rightarrow \infty$:

$$\begin{aligned} \phi_1^a(\mathbf{x}, z) &\sim z^{\Delta_{a,-}} \lambda^a(\mathbf{x}) + \dots \quad (\text{Poincare}) & \phi_1^a(t, \Omega, r) &\sim r^{-\Delta_{a,-}} \lambda^a(t, \Omega) \quad (\text{global}) \\ \phi_2^a(\mathbf{x}, z) &\sim z^{\Delta_{a,+}} \phi_0^a(\mathbf{x}) + \dots \quad (\text{Poincare}) & \phi_2^a(t, \Omega, r) &\sim r^{-\Delta_{a,+}} \phi_0^a(t, \Omega) \quad (\text{global}) \end{aligned} \quad (3.5)$$

A general solution to the linearized equations can be written (using Poincaré coordinates for definiteness) as:

$$\phi^a(\mathbf{x}, z) = \phi_1^a + \phi_2^a. \quad (3.6)$$

The solution ϕ_2^a is normalizable with respect to the standard Klein-Gordon norm in AdS spacetimes [7, 28, 29]. The solution ϕ_1^a is normalizable for $-4 \leq R_{\text{AdS}}^2 m^2 < -3$, while for $R_{\text{AdS}}^2 m^2 \geq -3$ it is not. As we describe below, the normalizable modes are candidates for propagating modes in AdS, while the non-normalizable modes (and normalizable modes when $R_{\text{AdS}}^2 m^2 < -3$) are candidate duals to perturbations of the Hamiltonian of the quantum system.

Higher-spin fields, such as the metric, will be dual to higher-spin operators on the boundary, such as the stress tensor. For these modes, a similar story about the boundary behavior applies. In the nonlinear supergravity theory, backreaction will couple metric modes to the scalar modes. If $m^2 < 0$ for the scalar masses, so that the dual operators are relevant, the backreaction will be such that the metric remains asymptotically anti-de Sitter, reflecting the conformal fixed point in the UV.

3.3 Quantum states in the CFT

Let us first consider the dynamics of the unperturbed CFT and its gravitational dual. Assume that we are working at low energies and at large N, λ in the dual field theory, so that the supergravity approximation in spacetime is valid.

Quantum states of the bulk supergravity at fixed time t can be represented via wavefunctionals of ϕ^a :

$$\Psi[\phi^a(t)] , \tag{3.7}$$

In the absence of boundary sources the states must be supported on field configurations that have boundary behavior specified by the second line in (3.5) (we will discuss the ambiguity for $(mR_{\text{AdS}})^2 \leq -3$ at the end of this section.) This can be seen for small fluctuations of the supergravity fields by building up the states via second quantization.

Because the AdS/CFT correspondence has a Hamiltonian version [23], the Hilbert spaces of the gauge theory and the dual string theory must be the same. Furthermore, one can define a Hamiltonian which has both a gauge theory and a string theory interpretation. The vacuum can be defined as the state preserving the $SO(4, 2)$ symmetry of the theory (the conformal group of the CFT or the isometry group of AdS.) The duality map for small fluctuations of the supergravity fields can be constructed explicitly by providing a map between the Fourier modes of the CFT operators \mathcal{O}^a and the creation and annihilation operators of the dual supergravity fields ϕ^a , as in refs. [7, 23, 30, 31]-[32].³

For states defined at past and future infinity, as in (3.1), we can push the fixed- t slices back to the far past and future, by defining spacelike hypersurfaces Σ_{\pm} at constant global or Poincaré time t_{\pm} , as shown in figures 1 and 2, and sending $t_{\pm} \rightarrow \pm\infty$. Note that in Poincaré coordinates, the $t_{\pm} \rightarrow \pm\infty$ limits of constant-time hypersurfaces are the unions of the horizons H_{\pm} and timelike infinity \mathbf{i}^{\pm} .

We are particularly concerned with the case that the supergravity states are semiclassical coherent states in the bulk,⁴ described by macroscopic expectation values for the field

³In the interacting, finite- N theory, this map must be modified, as pointed out in [32]. To our knowledge, the issues raised in that work have yet to be addressed.

⁴We use the term "coherent state" in the sense described by Yaffe [33]. In [34] it is stated that the large- N limit is not a classical limit in the usual sense, as it corresponds to a limit with a large number of fields. However, ref. [33] gives a precise definition of a classical limit, and gives a convincing if not complete set of arguments that the large N limit of a gauge theory is such a classical limit. The arguments are independent of the 't Hooft coupling, and matches our expectation that this limit is dual to the limit of classical string theory in anti-de Sitter space. However, we believe that the arguments in [33] require that one take $N \rightarrow \infty$ with fixed 't Hooft coupling λ . The limit in which g_{YM}^2 is fixed and small as $N \rightarrow \infty$ should be a different limit, dual to quantum string theory in ten-dimensional flat space.

operators $\phi^a(\mathbf{x}, z)$. At leading order in $1/N$, this expectation value satisfies the supergravity equations of motion and has the $z \rightarrow 0$ behavior of the normalizable modes. At fixed time t , such semiclassical coherent states are well-specified at this order in $1/N$ by a state in classical phase space: that is, by the value of the field $\phi^a = \phi_0^a$ and the field momentum $\pi_a \sim \dot{\phi}_a = \pi_{a,0}$ at fixed time t . The quantum wavefunctional

$$\Psi_{\phi_0^a, \pi_{a,0}}[\phi^a, t] \equiv \langle \phi^a | (\phi_0^a, \pi_{a,0}) \rangle \quad (3.8)$$

where $|\phi^a\rangle$ is an eigenstate of the field operator, is peaked at these values, with a width in phase space proportional to \hbar/N^2 .

Consider a solution $\phi^a(x, z, t)$ to the classical equations of motion with initial conditions $\phi_-^a \equiv \phi^a(t_-)$, $\pi_{a,-} \equiv \pi_a(t_-)$. If we fix the quantum state $|\psi_-\rangle$ at time t_- to be a coherent state $|(\phi_-^a, \pi_{a,-})\rangle$ with the wavefunctionals of ϕ^a and π_a peaked on these initial conditions at time t , then to leading order in $1/N$ the system will evolve in time through classical states peaked on $\phi^a(t), \pi_a(t)$. Now, let $|\psi_+(t_+)\rangle = |(\phi_+^a, \pi_{a,+})\rangle$. The transition amplitude

$$A = \langle \psi_+(t_+) | \psi_-(t_-) \rangle, \quad (3.9)$$

will be negligible at leading order in $1/N$ unless $(\phi_+^a, \pi_{a,+}) \sim (\phi^a(t_+), \pi_a(t_+))$ up to corrections of order $1/N$. Otherwise, there is no semiclassical trajectory contributing to A .

On the other hand, if the initial and final states are eigenstates $|\phi^a\rangle$ of the field operator, then the transition amplitude

$$A_{\text{pos}} = \langle \phi_+^a(t_+) | \phi_-^a(t_-) \rangle, \quad (3.10)$$

will generically receive contributions from semiclassical paths contributing to it for a range of ϕ_+^a, ϕ_-^a . These will be paths corresponding to solutions to the classical equations of motion, specified by the initial and final field values $\phi^a(t_{\pm}) = \phi_{\pm}^a$, as can be deduced from a stationary phase approximation. To make contact with the coherent state approach, one can use the fact that the coherent states form an overcomplete basis and may be used to construct a resolution of the identity [33]:

$$\mathbf{1} = \int D\phi^a D\pi_a |(\phi^a, \pi_a)(t)\rangle \langle (\phi^a, \pi_a)(t)|. \quad (3.11)$$

Let $\phi_n^a(t), \pi_{a,n}(t)$ be the classical positions and momenta at time $t \in [t_-, t_+]$ consistent with the initial and final conditions $\phi^a(t_{\pm}) = \phi_{\pm}^a$. The label n takes care of the cases where there may be more than one solution. If we insert (3.11) at time $t \in [t_-, t_+]$, we will find that at leading order in $1/N$, the dominant contributions will come from the coherent states specified by $\phi_n^a(t), \pi_{a,n}(t)$.

In order to make contact with the work of [7, 17], consider the case in (3.1) for which $|\psi_{\pm}\rangle$ are classical coherent states, consistent with a single classical trajectory $\phi^a(x, t)$. Then the matrix element of the operator in the dual CFT is specified by the boundary behavior of the expectation value of the bulk supergravity field:⁵

$$\langle \psi_+ | \mathcal{O}^a(\mathbf{x}) | \psi_- \rangle = \Delta_+ \phi_0(\mathbf{x}), \quad (3.12)$$

⁵For a certain class of operators this formula will receive corrections, as shown in [35].

(Note that if z has length dimension 1, (3.12) is dimensionally correct.) To leading order in the $1/N$ expansion, the classical "coherent" states are completely specified by the expectation values of classical operators [33]. The classical operators are the single-trace operators of the theory (they may be nonlocal in general.) At this order in $1/N$, every such operator is independent. For local single-trace operators, one must specify the expectation value for every frequency and spatial momentum. Alternatively, one may specify $\phi_0(x)$ in (3.12) for all x . In the dual theory, with in the linear approximation to the supergravity equations of motion, this specifies the classical solution completely. Note that general (highly quantum) states are not well-characterized by the one point functions, as discussed for example in [36].

To be more precise, we must consider the 5d metric coupled to the scalars. Therefore the classical coherent states are $|\phi^a, \pi_a; g^{\mu\nu}, \pi_{\mu\nu}\rangle$, and the eigenstates of the field operators are $|\phi^a, g^{\mu\nu}\rangle$. The topology and geometry of Σ_{\pm} is defined by the state. This topology can be rather different from a spacelike slice of AdS spacetime: for example, the spacetime may be an AdS-Schwarzschild black hole.

3.4 SUGRA field eigenstates and gravitational backreaction

The astute reader will worry about our use of eigenstates of the supergravity field operators. Such states have overlap with eigenstates of the Hamiltonian states of arbitrarily high energy, even for a mode of finite frequency. The point is that if π_n is the conjugate momentum for ϕ_n , the noninteracting Hamiltonian for this theory is:

$$H = \frac{1}{2}\pi_n^2 + \omega_n^2\phi_n^2 + \dots \tag{3.13}$$

If the uncertainty in ϕ_n vanishes, the uncertainty in π_n is infinite, and so the energy uncertainty is infinite. This is potentially disastrous when coupling the theory to gravity.

Nonetheless, we believe that we are safe so long as we calculate objects such as (3.10). First, in the dual CFT there is no gravity, and so there seems to be no problem in principle in considering states of as high an energy as we please. In the supergravity theory, a state of fixed but large energy, made up of normalizable modes, will not change the asymptotic structure of anti-de Sitter space. A black hole is the generic example of such a state, and black hole solutions do not disturb the asymptotic structure at timelike infinity.

Secondly, from the spacetime point of view, $|\phi\rangle$ is not a classical coherent state, and it makes no sense to simply insert $\langle\phi|T_{\mu\nu}|\phi\rangle$ into Einstein's equations in order to compute the backreaction. Instead, one can decompose this state via (3.11):

$$|\phi\rangle = \int D\phi_0 D\pi_0 |(\phi_0, \pi_0)\rangle \langle(\phi_0, \pi_0)|\phi\rangle \equiv \int D\phi_0 D\pi_0 |(\phi_0, \pi_0)\rangle \Psi_{(\phi_0, \pi_0)}(\phi) . \tag{3.14}$$

Here $\Psi_{\phi_0, \pi_0}(\phi)$ is the wavefunctional for a classical coherent state with the expectation values of ϕ, π peaked on ϕ_0, π_0 , and the peak has width $1/N$. The energy of such a state is also sharply peaked at its classical value. In the classical limit, one should compute the classical gravitational backreaction for each such classical state. In other words, (3.11) should really be considered as an integral over classical states of the scalars and the metric.

For each such classical state, time evolution will generate a classical spacetime. Some of these spacetimes will be strongly gravitating. Generically they will be black holes of arbitrarily high energy. But if we compute (3.10) for sufficiently weak fields, these highly energetic states will not contribute.

3.5 Perturbing the CFT

Next, let us consider the case when the $\mathcal{N} = 4$ SYM action is perturbed by local scalar operators. The case where the action is perturbed by the stress-tensor (which is dual to the bulk metric) will not be considered. The basic prescription is stated in [22, 23]: a perturbation of the CFT Hamiltonian by a (spacetime-dependent) coupling $\lambda^a(x)$ is dual to performing the path integral over supergravity modes $\phi^a(\mathbf{x}, r)$ with boundary conditions at timelike infinity specified by the first line of (3.5). In the classical, large- N limit, a given classical solution satisfying these boundary conditions will be a saddle point solution describing a transition amplitude between two coherent states.

3.5.1 The limit of linearized supergravity

Let us first consider scalar fields which remain small in the interior of the AdS spacetime. In this case bulk interactions can be neglected and a general solution $\phi^a(\mathbf{x}, z)$ with the $z \rightarrow 0$ behavior dominated by the first line of (3.5) can be written as:

$$\begin{aligned} \phi^a(x, z) &= \int d^4x' G_{\partial B}^a(x, z; x') \lambda^a(x') + \phi_v(z, x) \\ \phi_v(x, z) &\xrightarrow{z \rightarrow 0} z^{\Delta_+} \tilde{\phi}(x) . \end{aligned} \tag{3.15}$$

Here G is the bulk-boundary propagator in the AdS vacuum, as defined in [23] and more carefully in [37]. Note that even in the limit of linearized supergravity, the map between $\tilde{\phi}$ and the field eigenstates at Σ_{\pm} will depend on the coupling: this is because the first term on the right hand side of (3.15) has support on Σ_{\pm} . Therefore, to keep the states at Σ_{\pm} fixed while changing λ , we must also change $\tilde{\phi}$. On the other hand, changing the state will change $\tilde{\phi}$ and not λ .

The normalizable piece which scales as ϕ_2 in (3.5) is a linear combination of λ and $\tilde{\phi}$. In Poincaré coordinates, any linear combination is allowed when the momenta dual to \mathbf{x} is timelike, corresponding to the existence of propagating states for any such momentum. For spacelike momenta there are not propagating states, and the normalizable and non-normalizable modes must come in a specific linear combination in order to avoid a singularity in the interior of AdS [22, 23].

We should note that many singularities are resolvable in that they reflect interesting infrared physics in the dual quantum field theory, or D-brane sources in the interior of spacetime. However, it is important that not all such singularities are resolvable [38]. We leave this question, in the context of our discussion of holographic renormalization, for future work.

In the end, in addition to the piece of ϕ_2 which depends on λ , we may add a piece that is specified by $\tilde{\phi}$ in (3.15). (In global coordinates, the considerations above restrict

the frequency decomposition of $\tilde{\phi}$). This freedom has a reflection in the field theory: the one-point function at finite λ in the noninteracting, large- N limit is [7]:

$$\langle \mathcal{O}(\mathbf{x}) \rangle = \Delta_+ \tilde{\phi} + c \Delta_+ \int d^4 \mathbf{x}' \frac{\lambda(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2\Delta_+}}, \quad (3.16)$$

where c is a constant independent of $\lambda, \tilde{\phi}$, and we have chosen to state the results in Poincaré coordinates.

3.5.2 Perturbations in the interacting theory

We will be interested in relevant perturbations which grow in the infrared. For perturbations which become large in the IR the bulk interactions cannot be neglected and the spacetime can change drastically. This leads to two issues that we need to address.

First, for (3.1) to make sense as a generating functional for correlation functions, the states should be independent of variations of the couplings $\lambda^a(x)$. Given our experience with solutions to the linearized equations, we might be tempted to define the states via the expectation values of operators, which depends on the piece of the scalar fields behaving as ϕ_2 in (3.5), or via $\tilde{\phi}$ in (3.15). However, once we take nonlinearities of classical supergravity into account, the map between the quantum state and this data will depend on $\lambda(x)$.

These problems are avoided (in the supergravity approximation) by defining the states via wavefunctionals of ϕ^a (and $g^{\mu\nu}$) on spacelike slices Σ_{\pm} at fixed times t_{\pm} [21], so long as one only varies the couplings strictly in the interior of t_{\pm} . Since we have defined a set of states in the bulk in a way that is independent of the boundary conditions at timelike infinity, duality implies we have defined a set of states which are independent of the $4d$ couplings. However, describing these states as $4d$ field theory operators acting on the vacuum may be very difficult. To begin with, the geometry and topology of Σ_{\pm} may be very different from a spacelike slice of AdS. For example, upon perturbing the theory by a mass term, an "infrared wall" may develop at finite radius [38].⁶ In such a situation, the states must define a slice Σ_{\pm} of the appropriate topology, or there will be no semiclassical trajectory contributing to (3.1). Note that near the wall, one must specify more than the values of the supergravity fields in order to define the state: classical supergravity breaks down near such a wall, and the singularity is resolved by stringy and quantum effects.

3.6 The generating function of correlation functions

Given the prescription above, we can now construct all elements of (3.1). In the remainder of this paper, we wish to consider (3.1) as a solution to the classical Hamilton-Jacobi equations of the supergravity theory. In this formulation, the constants of motion will essentially be the couplings and the states. This map will make sense if, for fixed $|\psi_{\pm}\rangle$ and λ^a , there is a unique classical saddle point in the path integral representation of (3.1) for every small variation of λ^a . As we have stated above, this will not be true if we choose $|\psi_{\pm}\rangle$ to be definite classical coherent states. Instead, we will choose $|\psi_{\pm}\rangle = |\phi^a(t_{\pm}), g^{\mu\nu}(t_{\pm})\rangle$ to

⁶We would like to thank O. Aharony for correspondence on this issue.

be eigenstates of the field operators. In this case, the generating function of correlation functions in the large- N limit is the classical action

$$Z[\{\lambda\}] = \exp\left(\frac{i}{\hbar} S_{\text{cl}}[\lambda^a(\mathbf{x}), (\phi_-^a, g_-^{\mu\nu}), (\phi_+^a, g_+^{\mu\nu})]\right), \quad (3.17)$$

where S_{cl} is the classical supergravity action evaluated between Σ_{\pm} , on solutions to the classical equations of motion such that

$$\phi(\mathbf{x}, r \rightarrow \infty) \sim r^{-\Delta_{-,a}} \lambda^a(\mathbf{x}) \phi^a(t_{\pm}) = \phi_{\pm}^a, \quad g^{\mu\nu}(t_{\pm}) = g_{\pm}^{\mu\nu}, \quad (3.18)$$

and the metric is asymptotic AdS.

A small variation of (3.17) with respect to λ will lead to boundary terms at the timelike boundaries only. Since the value of the bulk fields is fixed at Σ_{\pm} , the variation there vanishes by construction. In §5 this point will be important in claiming that S_{cl} in (3.17) solves the Hamilton-Jacobi equation. For more general states we can compute correlation functions by integrating ϕ_{\pm} over some wavefunctionals which are sharply peaked on states of finite energy. This leads to the prescription in [21]. For example, one can suppress the high-energy fluctuations discussed in §3.4 by choosing the states to be described by smooth (*e.g.* gaussian) wavefunctionals peaked about the field eigenstates. In our proposal, one varies the Hamiltonian, and then one integrates over initial and final states in order to suppress the high-energy contributions. We are then assuming that the integral with respect to ϕ_{\pm} and the derivatives with respect to $\lambda(x)$ commute.

Before closing we must point out an additional subtlety in this discussion. In the range $-4 \leq m^2 < -3$, both $\phi_{1,2}$ in (3.5) are normalizable, and the identification of these with $\lambda^a, \tilde{\phi}^a$ can be reversed [7, 18]. The generating functions of correlation functions of the two theories are related by a Legendre transformation [18, 39]. In this work we will take the solution scaling as z^{Δ_-} to correspond to the coupling, although it is not always natural to do so (*c.f.* §2 of [17].)

4. Holographic renormalization and the Hamilton-Jacobi equation

In this section we will embark on a critical review and discussion of the results of [15, 16], in order to better explain and eventually answer the questions raised in the introduction. We will restate the results of those papers in some detail, as we will need to comment on some specific points.

4.1 General discussion

As pointed out by various authors, beginning with [23, 37], computations of correlators in both the bulk and boundary theories require regularization. The bulk calculations contain divergent terms in S_{SUGRA} arising from the $r \rightarrow \infty/z \rightarrow 0$ region of the spacetime. There is by now a well-defined procedure for subtracting these divergences in the bulk and interpreting this subtraction procedure as a choice of local ultraviolet counterterms in the dual field theory (see for example [40] for a review and references.) In the field theory,

the counterterms determine the beta functions of the theory. The result is a supergravity expression for the objects driving the RG flow in the dual field theory.

This suggests a regularized version of the AdS/CFT correspondence [14]. Consider the correspondence for the Poincaré patch of AdS spacetime. The classical Lagrangian of the bulk supergravity theory is integrated over $z > z_{UV}$, to define

$$S_{\text{reg}}(\phi_{UV}, g_{UV}; z_{UV}) = \int_{z \geq z_{UV}} d^5x \mathcal{L}(\phi, g), \quad (4.1)$$

where the Lagrangian is evaluated on solutions to the classical equations of motion, with boundary values

$$\phi_{UV}(\mathbf{x}) = \phi(\mathbf{x}, z_{UV}) ; \quad g_{UV, \mu\nu}(\mathbf{x}) = g_{\mu\nu}(\mathbf{x}, z_{UV}) . \quad (4.2)$$

As with the "unregulated" version of the correspondence, this prescription is well-defined in Euclidean space because the classical equations of motion with boundary conditions (4.2) have a unique nonsingular solution for $z > z_{UV}$.⁷ After subtracting a set of counterterms, S_{reg} is identified with the generating function of correlators in the dual theory, cut off at an energy scale proportional to z_{UV}^{-1} [2] (although it will correspond to a fairly complicated cutoff prescription [14].) In this picture, ϕ_{UV}, g_{UV} are dual to couplings in the cutoff theory. The evolution of the fields ϕ_{UV}, z_{UV} as one increases z_{UV} is expected to be dual to the renormalization group flow of the boundary couplings, as one lowers the UV cutoff, and it can be shown that the evolution of this generating function with z_{UV} is governed by a kind of Callan-Symanzik equation [7].

The radial evolution of $S_{\text{reg}}, \phi_{UV}$, and g_{UV} with z can be described by the Hamilton-Jacobi equation. In the asymptotic limit $z \rightarrow 0$, the radial Hamilton-Jacobi equation can be rewritten as a set of Callan-Symanzik equations for the boundary correlators [15, 16], via a construction we now review and discuss.

4.2 The radial Hamilton-Jacobi equation

In classical general relativity coupled to matter, the Hamiltonian constraint $\mathcal{H} = 0$ is a Hamilton-Jacobi equation of the form $K(\mathbf{a}, \mathbf{b}, t) = 0$ (2.5). The evolution of the fields ϕ, g can then be computed via either (2.2) or (2.4). We will use (2.2), and find that it has a close relationship to the RG equations of the dual field theory.

De Boer *et. al.* consider the Euclidean AdS/CFT correspondence. The Euclidean metric can be written using an ADM decomposition:

$$ds^2 = R_{\text{AdS}}^2 \mathcal{N}^2 dr^2 + R_{\text{AdS}}^2 g_{\mu\nu} (dx^\mu + \mathcal{N}^\mu dr) (dx^\nu + \mathcal{N}^\nu dr) , \quad (4.3)$$

where $\mathcal{N}, \mathcal{N}^\mu$ are the lapse and shift functions. Locally, we can use the diffeomorphism invariance to choose $\mathcal{N} = 1$ and $\mathcal{N}^\mu = 0$, and we will work in this gauge from now on. (In the Poincaré patch of AdS the metric in this gauge is related to (3.3) by $z = R_{\text{AdS}} e^{-r/R_{\text{AdS}}}$.) However, after gauge fixing we must still impose the equations of motion

⁷Again, some additional singular solutions are allowed in Euclidean space. These have definite physical interpretations and introduce no ambiguity in the interpretation of (4.1).

for \mathcal{N} and \mathcal{N}^μ . These give rise to the Hamiltonian constraint and the diffeomorphism constraints, respectively.

Note that we have put an explicit factor of R_{AdS}^2 in front. With this normalization, the factors of R_{AdS} in the classical supergravity action appear in the combinations $R_{\text{AdS}}/\ell_s = \lambda^{1/4}$ and $R_{\text{AdS}}/\ell_{p,5} = N^{2/3}$, where $\lambda = g_{\text{YM}}^2 N$ is the 't Hooft coupling. However, it means that $g_{\mu\nu}$ has mass dimension 2.

We write Hamilton's principal function

$$S[(g_{\mu\nu}(\mathbf{x}), \phi(\mathbf{x})), \mathbf{a}], \tag{4.4}$$

as a functional of the configuration space variables $(g_{\mu\nu}(\mathbf{x}), \phi(\mathbf{x}))$, and the constants of motion \mathbf{a} . In doing so we must take symmetry under diffeomorphisms into account. Two metrics which differ only by a coordinate transformation describes the same point in the configuration space. The diffeomorphism constraint:

$$\nabla^\mu \frac{\delta S}{\delta g^{\mu\nu}} + \frac{\delta S}{\delta \phi^a} \nabla_\nu \phi^a = 0, \tag{4.5}$$

ensures that S is invariant under 4-d coordinate transformations and therefore is a good function on the configuration space.

The Hamiltonian constraint $\mathcal{H} = 0$ is

$$\frac{1}{\sqrt{\hat{g}}} \left[\frac{1}{3} \left(g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} \right)^2 - g^{\mu\lambda} g^{\nu\rho} \frac{\delta S}{\delta g^{\mu\nu}} \frac{\delta S}{\delta g^{\lambda\rho}} - \frac{1}{2} G^{ab}(\phi) \frac{\delta S}{\delta \phi^a} \frac{\delta S}{\delta \phi^b} \right] = \mathcal{L}, \tag{4.6}$$

where

$$\mathcal{L} = \sqrt{\hat{g}} \left(\frac{1}{2} G_{ab}(\phi) \hat{g}^{IJ} \partial_I \phi^a \partial_J \phi^b + \hat{\mathcal{R}} - V(\phi) \right), \tag{4.7}$$

is taken to be the (bosonic) 5d Lagrangian for minimally coupled scalar fields ϕ^a and bulk 5d metric \hat{g}^{IJ} ($I, J = 1 \dots 5$). The constants of motion \mathbf{a} in (4.4) parametrize solutions to (4.6), (4.5); we will discuss them below. In a gravitational theory, S does not depend explicitly on r . Eq. (4.6) is the Hamilton-Jacobi equation in this context. Once we have solved for (4.6) for S , the equations of motion for g and ϕ following from Hamilton's equations are:

$$\begin{aligned} \frac{\partial \phi^a(\mathbf{x}, r)}{\partial r} &= \frac{G^{ab}(\phi)}{\sqrt{\hat{g}}} \frac{\delta S}{\delta \phi^b(\mathbf{x}, r)} \\ \frac{\partial g_{\mu\nu}(\mathbf{x}, r)}{\partial r} &= \frac{1}{\sqrt{\hat{g}}} \left(-2 \frac{\delta S}{\delta g^{\mu\nu}(\mathbf{x}, r)} + \frac{2}{3} g_{\mu\nu} g^{\lambda\rho} \frac{\delta S}{\delta g^{\lambda\rho}(\mathbf{x}, r)} \right). \end{aligned} \tag{4.8}$$

4.3 Solving the Hamilton-Jacobi equations

Let us consider the case with only marginal and relevant perturbations of the CFT, following [15, 16]. We will also follow these references and consider solutions S in the region $r \rightarrow \infty$, dual to the UV regime of the field theory. In that limit the authors of [15, 16] propose the following path to a solution. They write S and \mathcal{L} in a derivative expansion

$$S = S^{(0)} + S^{(2)} + \Gamma, \quad \mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)}, \tag{4.9}$$

where

$$\begin{aligned}
 S^{(0)} &= \int d^4x \sqrt{g} U(\phi) \\
 S^{(2)} &= \int d^4x \sqrt{g} \left(\Phi(\phi) \mathcal{R} + \frac{1}{2} M_{ab}(\phi) g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b \right) \\
 \mathcal{L}^{(0)} &= -\sqrt{\hat{g}} V \\
 \mathcal{L}^{(2)} &= \sqrt{\hat{g}} \left(\hat{\mathcal{R}} + \frac{1}{2} G_{ab} \hat{g}^{IJ} \partial_I \phi^a \partial_J \phi^b \right). \tag{4.10}
 \end{aligned}$$

Here \mathcal{R} is the 4d Ricci curvature for g and (U, M_{ab}, Φ) are some functions of $(\phi(\mathbf{x}), g_{\mu\nu}(\mathbf{x}))$ to be determined from (4.6). Γ is the nonlocal part of S . Note that in this discussion we are assuming that the spacetime effective action \mathcal{L} is captured by the zero- and two-derivative terms. This is dual to the statement that we work to leading order in $1/N$ and $\lambda^{-1/4}$.

Next, one solves (4.6) order by order in the derivative expansion. If we define

$$\{A, B\} \equiv \frac{1}{\sqrt{g}} \left[\frac{1}{3} \left(g^{\mu\nu} \frac{\delta A}{\delta g^{\mu\nu}} \right) \left(g^{\lambda\rho} \frac{\delta B}{\delta g^{\lambda\rho}} \right) - g^{\mu\lambda} g^{\nu\rho} \frac{\delta A}{\delta g^{\mu\nu}} \frac{\delta B}{\delta g^{\lambda\rho}} - \frac{1}{2} G^{ab} \frac{\delta A}{\delta \phi^a} \frac{\delta B}{\delta \phi^b} \right], \tag{4.11}$$

then the Hamilton-Jacobi equation breaks up into a set of equations for each order in the derivative expansion. These are written in [15, 16] as:

$$\begin{aligned}
 \{S^{(0)}, S^{(0)}\} &= \mathcal{L}^{(0)} \\
 2 \{S^{(0)}, S^{(2)}\} &= \mathcal{L}^{(2)} \\
 2 \{S^{(0)}, \Gamma\} + \{S^{(2)}, S^{(2)}\} &= 0. \tag{4.12}
 \end{aligned}$$

This scheme for solving the Hamilton-Jacobi equations via a derivative expansion is sensible as $r \rightarrow \infty$ because the various terms in (4.9) scale differently in this limit: terms with lower number of derivatives diverge more rapidly. The result is a clean interpretation of S . $S^{(0)}$, $S^{(2)}$ correspond to divergent counterterms; a study of explicit solutions to (4.12) shows that they are local. Γ is the spacetime effective action minus these counterterms, that is, the regularized generating function of correlators in the dual field theory.⁸

This procedure for solving (4.6) is adapted to the limit $r \rightarrow \infty$. Furthermore, following [15, 16], we have ignored the terms $\{S^{(2)}, \Gamma\}$ and $\{\Gamma, \Gamma\}$ that appear in the full Hamilton-Jacobi equations, as they are negligible in the $r \rightarrow \infty$ limit. An extension of the formalism to finite r is desirable if one wishes to study trajectories of renormalization group flows over a range of scales. We will return to this issue in §5.5.

Even in this limit, there is an additional interpretational problem with this method for solving the Hamilton-Jacobi equations. As we discussed in §2, there are many possible solutions to the equations of motion, corresponding to different choices of constants of motion \mathbf{a} . On the other hand, if we desire solutions which are nonsingular in the interior

⁸The second term in the last line of (4.12) corresponds to the gravitational anomaly in four dimensions, and the first term includes the expectation value of the trace of the stress tensor. Therefore it makes sense to assign Γ dimension 4 in the derivative expansion. For a generalization to other dimensions see [19].

$r < \infty$ of Euclidean AdS spacetimes, no such freedom exists [23]. The constants of motion \mathbf{a} represent the states in the Lorentzian correspondence. The QFT dual of this will become apparent in §5. Correlation functions in Euclidean space rotate to vacuum correlators: in other words, a particular state is selected.

The classical action (4.1) will solve the Hamilton-Jacobi equation and generate solutions which are nonsingular in the interior. For the formalism of [15, 16] to match the results of [22, 23], it must be true that as $r \rightarrow \infty$ ($z_{UV} \rightarrow 0$), (4.1) can be written as (4.9) and is a solution to (4.12).

All of this said, other solutions to the Hamilton-Jacobi equations exist, even in the case of Euclidean signature. The solution to these equations will not have an interpretation as the generating function of correlation functions. We will discuss one such solution, and its interpretation, in §5.4.

4.4 Relation to the renormalization group

4.4.1 Beta functions

Near the boundary $r \rightarrow \infty$ the bulk fields behave as $\phi^a \sim e^{-\Delta-r}$ and are dual to the coupling constants of the field theory. The evolution of ϕ with r should be related to the running of the dual coupling under the renormalization group. The bulk fields are in general spacetime dependent, so we should work with spacetime-dependent couplings [41, 42]. However, if the couplings are slowly varying in \mathbf{x} , then they can be treated as constant in the UV, and we should recover the standard RG equations in that limit. The bulk dual of this statement is that as $r \rightarrow \infty$, $S^{(0)}$ dominates over the higher-derivative and nonlocal terms in S . Therefore, let us first consider solutions which are constant in \mathbf{x} .

The first equation in (4.12) is

$$V = \frac{1}{3}U^2 - \frac{1}{2}G^{ab}\partial_a U \partial_b U, \tag{4.13}$$

and determines U in terms of V (up to terms of total dimension 4). Eq. (4.8) then becomes

$$\begin{aligned} \partial_r \phi^a &= G^{ab} \partial_b U \\ \partial_r g_{\mu\nu} &= -\frac{1}{3}U(\phi)g_{\mu\nu}, \end{aligned} \tag{4.14}$$

where both $\partial_r \phi$ and $\partial_r g$ have corrections which can be neglected near the boundary. The second equation in (4.14) can be solved with the ansatz $g_{\mu\nu}(r, \mathbf{x}) = \rho^2(r, \mathbf{x})\tilde{g}_{\mu\nu}(\mathbf{x})$. Here \tilde{g} is dimensionless and independent of r , while the scale factor ρ has mass dimension 1 and satisfies

$$\partial_r \ln(R_{\text{AdS}}\rho) = -\frac{1}{6}U(\phi). \tag{4.15}$$

The study of bulk probes in the AdS/CFT correspondence shows that the rescaling in the boundary theory is simply related to the rescaling of g in the bulk. Furthermore, since the solutions ϕ_1 in (3.5) dominate as $r \rightarrow \infty$, we can identify the coupling λ^a at some UV scale ρ with the dual field $\phi(r(\rho))$, up to a power of ρ .⁹ Following these two observations, de

⁹If $m^2 R_{\text{AdS}}^2 \leq -3$ then this is not always true, as noted in §2.2. However, in Euclidean space, ϕ_1 and ϕ_2 are proportional in Fourier space, and related by convolution with the boundary Green function in position space (*c.f.* [18].)

Boer *et. al.* propose that the beta function be defined as:

$$\beta^a \equiv \rho \partial_\rho \phi^a = \frac{1}{\partial_r \ln(\rho)} \partial_r \phi^a = -\frac{6}{U(\phi)} G^{ab} \partial_b U(\phi). \quad (4.16)$$

Because $\phi \sim z^{\Delta - a} \lambda_a(\mathbf{x})$ is dimensionless, we identify β^a as the beta functions for the dimensionless coupling. (The difference between this and the beta function for λ is that the latter will have no linear term.)

Let us study the solutions to these equations in more detail. The spacetime effective potential has the form

$$V = 12 - \frac{1}{2} m_a^2 (\phi^a)^2 + g_{abc} \phi^a \phi^b \phi^c. \quad (4.17)$$

If $G_{ab} = \eta_{ab} + O(\phi^2)$ then the solution for U is given by [15, 16]:

$$U = -6 - \frac{1}{2} \vartheta_a \phi^a \phi^a + \frac{g_{bc}^a}{8 - \Delta_a - \Delta_b - \Delta_c} \phi^a \phi^b \phi^c. \quad (4.18)$$

(Note that in [15, 16] the sign in front of the first term of the right hand side of (4.18) is opposite to that in (4.17). We have checked that the signs here are self consistent.) Here ϑ_a is related to m_a^2 by

$$\vartheta_a^2 - 4\vartheta_a = m_a^2. \quad (4.19)$$

Choosing the root $\vartheta_a = 4 - \Delta_a$, leads to the beta function:¹⁰

$$\beta^a = -(4 - \Delta_a) \phi^a - \frac{g_{bc}^a}{8 - \Delta_a - \Delta_b - \Delta_c} \phi^b \phi^c. \quad (4.20)$$

If the OPE coefficients

$$\mathcal{O}_b(x) \mathcal{O}_c(y) \sim \frac{C_{bc}^a}{|x - y|^{\Delta_b + \Delta_c - \Delta_a}} \mathcal{O}_a(y), \quad (4.21)$$

are equal to

$$C_{bc}^a = -\frac{2g_{bc}^a}{S_3 (8 - \Delta_a - \Delta_b - \Delta_c)}, \quad (4.22)$$

where S_3 is the volume of a unit 3-sphere, then (4.20) is precisely of the form derived, for constant couplings, in for example [43]. This is an unsurprising answer, as we expect the three-point functions in the bulk to be related to the boundary OPEs.

The actual relationship between g_{abc} and the boundary OPE coefficients was calculated in [37]. They differ by a complicated ratio of gamma functions. However, unless the OPEs in question are "resonant", such that $\Delta_b + \Delta_c - \Delta_a = 4$, the quadratic term in the beta functions are scheme dependent. In fact, standard RG schemes such as minimal subtraction will lead to quadratic terms in the beta function only if there are associated divergences, which happens when $\Delta_b + \Delta_c - \Delta_a \geq 4$ in (4.21).¹¹ At best we only expect universal answers at quadratic order in λ and to zeroth order in $4 - \Delta_a = \epsilon_a \ll 1$. Our conclusion is

¹⁰In §5.4 we will discuss the other root of (4.19).

¹¹See for example [44] for an extensive discussion of the scheme dependence of conformal perturbation theory in two dimensions. Most of the basic lessons lift to higher dimensions.

that the holographic RG calculation outlined in [15, 16] correspond to a particular choice of scheme that is closer to the schemes used in refs. [43, 45]. These schemes are natural and useful in conformal perturbation theory; in particular they are useful for studying the approach to nontrivial infrared fixed points.

4.4.2 The Callan-Symanzik equation

According to the AdS/CFT correspondence (3.1),

$$\Gamma = \ln(Z) \quad (4.23)$$

is the generating function of connected correlation functions, in the limit $r \rightarrow \infty$ or $z \rightarrow 0$, if we assume that $S = S_{\text{SUGRA}}$. According to de Boer *et. al.*, the asymptotic correlation functions are:

$$\langle O_{a_1}(\mathbf{x}_1) O_{a_2}(\mathbf{x}_2) \dots O_{a_n}(\mathbf{x}_n) \rangle_c \stackrel{?}{=} \frac{1}{\sqrt{g(\mathbf{x}_1)}} \frac{\delta}{\delta \phi^{a_1}(\mathbf{x}_1)} \dots \frac{1}{\sqrt{g(\mathbf{x}_n)}} \frac{\delta}{\delta \phi^{a_n}(\mathbf{x}_n)} \Gamma, \quad (4.24)$$

where $\langle \dots \rangle_c$ is the connected piece of the correlator. However, with our normalization, ϕ is dimensionless and so $\delta/\delta\phi(x)$ has mass dimension d . $g_{\mu\nu}$ has mass dimension 2, so that the right hand side above is dimensionless. If we wish O_{a_k} to correspond to operators of dimension Δ_{+,a_k} , we need to modify this expression. We conjecture that the correct asymptotic expression is:

$$\begin{aligned} \langle O_{a_1}(\mathbf{x}_1) O_{a_2}(\mathbf{x}_2) \dots O_{a_n}(\mathbf{x}_n) \rangle_c &= \frac{\rho^{\Delta_{+,a_1}}}{\sqrt{g(\mathbf{x}_1)}} \frac{\delta}{\delta \phi^{a_1}(\mathbf{x}_1)} \dots \frac{\rho^{\Delta_{+,a_n}}}{\sqrt{g(\mathbf{x}_n)}} \frac{\delta}{\delta \phi^{a_n}(\mathbf{x}_n)} \Gamma \\ &= \frac{\rho^{-\Delta_{-,a_1}}}{\sqrt{\tilde{g}(\mathbf{x}_1)}} \frac{\delta}{\delta \phi^{a_1}(\mathbf{x}_1)} \dots \frac{\rho^{-\Delta_{-,a_n}}}{\sqrt{\tilde{g}(\mathbf{x}_n)}} \frac{\delta}{\delta \phi^{a_n}(\mathbf{x}_n)} \Gamma. \end{aligned} \quad (4.25)$$

As evidence, we will find below that the Hamilton-Jacobi equations imply the statement that the left hand side of (4.25) satisfies the Callan-Symanzik equation. Furthermore, $\rho \rightarrow 1/z$ as $z \rightarrow 0$, and $\phi^a \rightarrow z^{\Delta_{-,a}} \lambda^a$. Therefore, using (3.5), we can rewrite (4.25) as:

$$\langle O_{a_1}(\mathbf{x}_1) O_{a_2}(\mathbf{x}_2) \dots O_{a_n}(\mathbf{x}_n) \rangle_c = \frac{1}{\sqrt{\tilde{g}(\mathbf{x}_1)}} \frac{\delta}{\delta \lambda^{a_1}(\mathbf{x}_1)} \dots \frac{1}{\sqrt{\tilde{g}(\mathbf{x}_n)}} \frac{\delta}{\delta \lambda^{a_n}(\mathbf{x}_n)} \Gamma, \quad (4.26)$$

which is the expected definition of the correlation functions.¹²

The third equation in (4.12) essentially is a local form of the Callan-Symanzik equation [41, 42]. Using the above result, it becomes:

$$\frac{1}{Z} \left[\rho(\mathbf{x}) \frac{\delta}{\delta \rho(\mathbf{x})} + \beta^a(\mathbf{x}) \frac{\delta}{\delta \phi^a(\mathbf{x})} \right] Z = -\frac{6\sqrt{g}}{U(\phi(\mathbf{x}))} \left\{ S^{(2)}(\mathbf{x}), S^{(2)}(\mathbf{x}) \right\}. \quad (4.27)$$

The right hand side is the contribution of the conformal anomaly [19, 46, 47]. Note that (4.27) is the asymptotic (and therefore local) form of the CS equation. For finite r , (4.27) becomes non-local.

¹²Eq. (4.26) also follows from rewriting a general conformal correlation function in a diffeomorphism invariant way using the metric $g_{\mu\nu}$ instead of $\tilde{g}_{\mu\nu}$ (which corresponds to (4.24)) and then plugging the relation $g_{\mu\nu} = \rho^2(r) \tilde{g}_{\mu\nu}$.

By varying (4.27) with respect to $\phi^a(\mathbf{x})$ and using (4.25), we arrive at a local form of the CS equations:

$$\left[\rho(\mathbf{x}) \frac{\delta}{\delta \rho(\mathbf{x})} + \sum_b \beta^b(\mathbf{x}) \frac{\delta}{\delta \phi^b(\mathbf{x})} \right] \langle O_{a_1}(\mathbf{x}_1) \dots O_{a_n}(\mathbf{x}_n) \rangle_c - \sum_{k=1}^n \int d^d x \gamma_{a_k}^{b_k}(\mathbf{x} - \mathbf{x}_k) \langle O_{a_1}(\mathbf{x}_1) \dots O_{b_k}(\mathbf{x}) \dots O_{a_n}(\mathbf{x}_n) \rangle_c = 0, \quad (4.28)$$

where

$$\gamma_a^b(\mathbf{x} - \mathbf{x}_b) = -(4 - \Delta_a) \delta_a^b \delta(\mathbf{x} - \mathbf{x}_a) + \frac{\delta}{\delta \phi_a(\mathbf{x}_a)} \beta^b(\phi(\mathbf{x})). \quad (4.29)$$

This definition of the anomalous dimension gives the deviation of the operator dimension from that at the UV fixed point, rather than the deviation from that of a free scalar.¹³

As pointed out in [42], (4.28) can be thought of as the Callan-Symanzik equation for spacetime-dependent couplings, first described by Osborn [41]. The beta functions β^a are the coefficients of the trace of the stress tensor: $\Theta(x) = -\sum_a \beta^a \mathcal{O}_a$. We can transform this equation into one which describes the behavior of Γ_n as one rescales all of its arguments \mathbf{x}_i . The essential point is that if a function f which itself has mass dimension Δ is a function of n variables with definite mass dimension, an infinitesimal rescaling of any k of these variables can be traded for an infinitesimal rescaling of the other $n - k$ variables plus an overall rescaling of f . This statement holds in the presence of additional constraints on the variables, so long as one imposes the constraints at the end. Write the n -point function in (4.25) as:

$$\begin{aligned} \Gamma_n &\equiv \langle \mathcal{O}_{a_1}(\mathbf{x}_1) \dots \mathcal{O}_{a_n}(\mathbf{x}_n) \rangle \\ &= \Gamma_n(\mathbf{x}_i, \rho(\mathbf{x}), \tilde{g}_{\mu\nu}(\mathbf{x}), \phi(\mathbf{x})) \end{aligned} \quad (4.30)$$

where Γ is taken to be a nonlocal functional of ρ, \tilde{g} , and ϕ . Note that there are additional dimensionful parameters implied by the spacetime dependence of ρ, \tilde{g} , and ϕ . Therefore we can relate a rescaling of ρ itself for a rescaling of \mathbf{x}_i and the variation of the ρ, \tilde{g} , and ϕ under a rescaling of the coordinates they depend on.

In the end, if we integrate (4.28) over \mathbf{x} and apply the arguments above, we find that:

$$\begin{aligned} &\sum_{k=1}^n \mathbf{x}_k \cdot \frac{\partial}{\partial \mathbf{x}_k} \langle O_{a_1}(\mathbf{x}_1) \dots O_{a_n}(\mathbf{x}_n) \rangle_c \\ &+ \int d^4 y \left[\sum_b \left(-\mathbf{y} \cdot \frac{\partial}{\partial \mathbf{y}} \phi^b(\mathbf{y}) + \beta^b(\phi) \right) \frac{\delta}{\delta \phi^b(\mathbf{y})} \right] \langle O_{a_1}(\mathbf{x}_1) \dots O_{a_n}(\mathbf{x}_n) \rangle_c \\ &- \int d^4 y \left[\left(\mathbf{y} \cdot \frac{\partial}{\partial \mathbf{y}} \rho(\mathbf{y}) \right) \frac{\delta}{\delta \rho(\mathbf{y})} + \delta \tilde{g}_{\mu\nu}(\mathbf{y}) \frac{\delta}{\delta \tilde{g}_{\mu\nu}(\mathbf{y})} \right] \langle O_{a_1}(\mathbf{x}_1) \dots O_{a_n}(\mathbf{x}_n) \rangle_c + \\ &\sum_{n=1}^k \left(\delta_{a_k}^{b_k} \Delta_{a_k} - \tilde{\gamma}_{a_k}^{b_k}(\phi) \right) \langle O_{a_1}(\mathbf{x}_1) \dots O_{b_k}(\mathbf{x}_k) \dots O_{a_n}(\mathbf{x}_n) \rangle_c = 0. \end{aligned} \quad (4.31)$$

¹³Note that in [15, 16], the term $4 - \Delta$ is missing from γ . It comes from commuting $2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}}$ through $\frac{\rho}{\sqrt{g}} \frac{\delta}{\delta \phi}$.

where

$$\tilde{\gamma}_a^b = -(4 - \Delta_a)\delta_a^b + \nabla_a\beta^b \tag{4.32}$$

is the anomalous dimension matrix, and the variation of \tilde{g} under rescaling is:

$$\delta\tilde{g}_{\mu\nu} = \mathbf{x} \cdot \partial_{\mathbf{x}}\tilde{g}_{\mu\nu} - x^\lambda\partial_\mu\tilde{g}_{\lambda\mu} - x^\lambda\partial_\nu\tilde{g}_{\mu\lambda} . \tag{4.33}$$

If ϕ and ρ are constant, eq. (4.31) is the standard form of the renormalization group equations, which follows directly from the Ward identities for broken scale invariance (see for example [44, 45].) The beta functions are the coefficients of the trace of the stress tensor: $\Theta(\mathbf{x}) = -\sum_a \rho^{\Delta-a}\beta^a\mathcal{O}_a$ (where the factor of ρ ensures that Θ has dimension d .) In §5.2, we will show by direct calculation in the perturbed CFT that the additional terms in (4.31) for spacetime dependent ϕ, ρ in (4.31) must also appear.

4.4.3 Caveats

We close this section with two comments. First, in discussions of the holographic renormalization group [14, 15]-[16], it is assumed that (4.25) at finite z is a good definition of the correlation functions in a theory cut off at the energy scale $\Lambda = 1/z$. This statement requires some care. As we will discuss in §5.5, there is not a direct relationship between $\phi(\mathbf{x}, z_{UV})$ and the coupling $\lambda(\mathbf{x})$ at finite z_{UV} : rather, ϕ is determined by both the coupling and the state of the theory. In the Euclidean calculations, ϕ will be a function of the couplings; we can therefore imagine a scheme in which ϕ is the coupling, although this may be related to more standard schemes by a complicated and possibly nonanalytic redefinition of the couplings.

Secondly, the choice of solution to the Hamilton-Jacobi equations for the right hand side of (4.25) changes the interpretation. In general the solution S is a function of constants of motion \mathbf{a} (as written in (4.4)) . When one takes the derivatives in (4.24), one is varying over classical solutions with the constants \mathbf{a} held fixed. The result depends on the definition as well as the numerical value of the constants of motion. On the other hand, the Euclidean correlators should be uniquely determined by the couplings. The assumption here, and in other works on holographic renormalization, is that the right hand side of (4.25) should be the nonlocal part of S_{reg} in (4.1).

5. Holographic RG in Lorentzian signature

In this section we will generalize the results of [15, 16] and of §4 to the Lorentzian version of the AdS/CFT correspondence. The essential difference is that normalizable, nonsingular solutions to the equations of motion exist, so that the coefficient λ^a defining the asymptotic behavior $\phi^a \rightarrow_{z \rightarrow 0} z^{\Delta-a}\lambda^a$ no longer unambiguously defines solutions to the equations of motion. The dual of this statement is that correlation functions in the perturbed CFT will depend on both the couplings and the state of the field theory [7, 17].

This leads to our solution to the main question posed in the introduction. Solving the second-order equations of motion for supergravity fields via the Hamilton-Jacobi method involves two steps. The first step is to solve the Hamilton-Jacobi equations. The solutions

are functions of the values of the fields at fixed z ; the functional form will depend on the state of the system. The second step is to solve (the first-order) Hamilton's equations, which requires that one specifies the values of the fields at fixed z . In the field theory dual, the first step is dual to solving the Callan-Symanzik equations, and to computing the beta functions. We will see that the Callan-Symanzik equations and their solutions are modified by the choice of state in precisely the same way that the Hamilton-Jacobi equation is. The second step is roughly dual to solving the first-order equations $\Lambda \partial_\Lambda \lambda^a = \beta^a$ specifying the flow of the couplings with scale. (We say roughly because the precise duality defines a flow of a combination of the couplings and the one-point functions of the associated operators, as we will discuss.)

Thus, despite the apparent first-order nature of the RG equations, they contain all of the information needed to specify any desired solution to the dual supergravity equations of motion. In §5.1-§5.3 we show this in detail in the limit $z \rightarrow 0$. In §5.4 we discuss a different class of solutions to the Hamilton-Jacobi equations and discuss their interpretation in terms of a quantum field theory with an IR cutoff. In §5.5 we discuss the extension of holographic RG deep into the infrared of the field theory. In §5.6 we discuss the bulk picture of Wilsonian renormalization.

5.1 The Hamilton-Jacobi functional in Lorentzian signature

As in §4.2, the spacetime equations of motion can be solved by specifying the fields at some constant radial coordinate — r in (3.2) and z in (3.3) — and solving the radial Hamilton-Jacobi equations. Any solution to the Hamilton-Jacobi equation S , corresponds to a choice of some constant of motion \mathbf{a} . As in §4, one can try to solve the HJ equation with the on-shell action S_{SUGRA} . However, for a general constant of motion \mathbf{a} , by varying S_{SUGRA} along the set of classical solutions with fixed \mathbf{a} , the expression $\pi(\phi) = \frac{\delta S}{\delta \phi}$ for the classical momentum fails due to boundary terms at Σ_\pm (see figures 1 and 2).¹⁴ S_{SUGRA} will solve the HJ equation if and only if there is a choice of constant of motion \mathbf{a} for which the variation of S_{SUGRA} do not produce boundary terms at Σ_\pm . As explained in §3, that is exactly the case when the state is held fixed. We therefore conclude that the on-shell SUGRA action cut off at some distance from the boundary

$$S(\phi_{\text{UV}}, g_{\text{UV}}^{\mu\nu}, r_{\text{UV}}; \mathbf{a}) = \int^{r_{\text{UV}}} dr d^4x \sqrt{g} \mathcal{L}(\phi, g^{\mu\nu}), \quad (5.1)$$

evaluated on a family of classical solutions interpolating between fixed initial and final states, solves the bulk radial Hamilton-Jacobi equation. Furthermore, (5.1) generates the boundary correlation functions cut off at some scale $l(r_{\text{UV}})$:

$$\frac{1}{\sqrt{g(\mathbf{x}_1)}} \frac{\delta}{\delta \lambda^{a_1}(\mathbf{x}_1)} \cdots \frac{1}{\sqrt{g(\mathbf{x}_n)}} \frac{\delta}{\delta \lambda^{a_n}(\mathbf{x}_n)} S = \langle \psi_+ | T[\mathcal{O}_{a_1}(\mathbf{x}_1) \cdots \mathcal{O}_{a_n}(\mathbf{x}_n)] | \psi_- \rangle_c. \quad (5.2)$$

Here ψ_\pm denote the eigenvalues of the scalar fields and the metric at times t_\pm , and the subscript c denotes the connected correlation functions. In these equations \mathbf{a} represents

¹⁴This boundary is a spacelike slice at any $r(z)$ in global (Poincaré) coordinates. In Poincaré coordinates, in the limit $t_\pm \rightarrow \pm\infty$, these boundaries include the $z \rightarrow \infty$ surface $H_- \cup H_+$ as well.

the data which specifies the eigenstates of the bulk fields at Σ_{\pm} . $\lambda^a(\mathbf{x})$ are the dimensionful couplings. In §5.5 we will argue that the relationship between ϕ and λ are nontrivial; near the boundary of AdS, however, they will be related by a power of the scale factor of the metric, as in §4.

The constant of motion \mathbf{b} is given by (2.4). Since the classical solution is uniquely determined by the initial and final states and the UV couplings [21], the constant of motion \mathbf{b} is a function of the states and the UV coupling (and does not depend on the radial coordinate.)

We are now close to a solution of problem (1) in §1. Next, in §5.2 we will write the Callan-Symanzik equation for matrix elements of time-ordered products between arbitrary states. Finally, in §5.3 we will show that the same Callan-Symanzik equations arises as part of the Hamilton-Jacobi equations for S . This will complete our solution.

For the remainder of this section we will stick to Poincaré coordinates for simplicity's sake.

5.2 The Callan-Symanzik equation for nontrivial matrix elements

As we have just intimated, the bulk data which completely specify a solution to the supergravity equations of motion are dual to the couplings and state of the field theory. The non-vacuum states are the new ingredient in the Lorentzian version of the correspondence. To understand holographic renormalization, we must therefore derive the Callan-Symanzik equation for general matrix elements of time-ordered products of operators:¹⁵

$$C_{n;\pm} = \langle \psi_+(t_+) | T [\mathcal{O}_1(\mathbf{x}_1) \dots \mathcal{O}_n(\mathbf{x}_n)] | \psi_-(t_-) \rangle, \quad (5.3)$$

for d -dimensional conformal field theories perturbed by the interaction Hamiltonian

$$S_{\text{int}} = \sum_a \int_{t=t_-}^{t=t_+} d^d x \epsilon^{\Delta_a - d} u^a(\mathbf{x}) \mathcal{O}_a(\mathbf{x}), \quad (5.4)$$

where \mathcal{O}_a are marginal and relevant operators of dimension Δ_a , ϵ is a cutoff scale with dimensions of length, and u^a are spacetime-dependent dimensionless couplings (as in [41].) We will assume that the operators \mathcal{O}_k in (5.3) are scalar operators of definite dimension Δ_k at the UV fixed point and that the background is flat d -dimensional Minkowski space. In general, however, the spacetime dependence of the couplings in (5.4) means that couplings to nonscalar operators will be generated along the RG flow. The sum in (5.4) should be taken to include these couplings. A more elegant treatment would be to consider the couplings to higher-spin operators as background gauge, metric, and tensor fields after the fashion of [41]. We leave this for future work.

The starting point is the statement that:

$$\begin{aligned} & \sum_k \left(\mathbf{x}_k \cdot \partial_{\mathbf{x}_k} + \hat{D}_k \right) \langle \psi_+ | T (\mathcal{O}_1(\mathbf{x}_1) \dots \mathcal{O}_k(\mathbf{x}_k) \dots \mathcal{O}_n(\mathbf{x}_n)) | \psi_- \rangle \\ &= i \sum_k \langle \psi_+ | T (\mathcal{O}_1(\mathbf{x}_1) \dots [Q(t_k), \mathcal{O}_k(\mathbf{x}_k)] \dots \mathcal{O}_n(\mathbf{x}_n)) | \psi_- \rangle \end{aligned} \quad (5.5)$$

¹⁵We would like to thank C. Beasley and H. Schnitzer for discussions about this derivation.

Here $Q(t_k) = \int J^0$ is the charge corresponding to the scale current $J^\nu = x^\mu T_\mu^\nu$:

$$Q(t_k) = \int_{t=t_k} d^{d-1} x x^\mu T_\mu^0 \quad (5.6)$$

with $T_{\mu\nu}$ the stress tensor. \hat{D} is the dilatation operator defined by

$$\mathcal{O}_a(\mathbf{x} + \lambda \mathbf{x}) = \mathcal{O}_a(\mathbf{x}) + \lambda \mathbf{x} \cdot \partial_{\mathbf{x}} \mathcal{O}_a + \lambda \hat{D} \mathcal{O}_a(\mathbf{x}) + \dots, \quad (5.7)$$

and the subscript on \hat{D} in (5.5) indicates which of the \mathcal{O}_a it acts on.

Now

$$\begin{aligned} & \langle \psi_+ | T(\mathcal{O}_1 \dots (\mathcal{O}_k Q(t_k) \mathcal{O}_{k+1} - \mathcal{O}_k Q(t_{k+1}) \mathcal{O}_{k+1}) \dots \mathcal{O}_n) | \psi_- \rangle \\ &= \int_{t_{k+1}}^{t_k} \langle \psi_+ | T(\mathcal{O}_1 \dots \mathcal{O}_k \partial_0 Q(t) \mathcal{O}_{k+1} \dots \mathcal{O}_n) | \psi_- \rangle, \end{aligned} \quad (5.8)$$

which we can combine with (5.5) to write:

$$\begin{aligned} & \sum_k \left(\mathbf{x}_k \cdot \partial_{\mathbf{x}_k} + \hat{D}_k \right) \langle \psi_+ | T(\mathcal{O}_1(\mathbf{x}_1) \dots \mathcal{O}_k(\mathbf{x}_k) \dots \mathcal{O}_n(\mathbf{x}_n)) | \psi_- \rangle \\ &= -i \int_{t_-}^{t_+} dt \langle \psi_+ | T(\partial_t Q(t) \mathcal{O}_1 \dots \mathcal{O}_n) | \psi_- \rangle \\ &+ i \langle \psi_+ | Q(t_+) T(\mathcal{O}_1 \dots \mathcal{O}_n) | \psi_- \rangle - i \langle \psi_+ | T(\mathcal{O}_1 \dots \mathcal{O}_n) Q(t_-) | \psi_- \rangle. \end{aligned} \quad (5.9)$$

Note the presence of the two extra boundary terms in the last line. These vanish when $|\psi_- \rangle = |\psi_+ \rangle = |0\rangle$, where $|0\rangle$ is the scale-invariant vacuum state.

We can rewrite $\int dt \partial_t Q(t) = \int d^d x \partial_t (x^\mu T_\mu^0)$ via integration by parts:

$$\begin{aligned} \int d^d x \partial_t (x^\mu T_\mu^0) &= \int d^d x (T_0^0 + x^\mu \partial_0 T_\mu^0) = \int d^d x (T_0^0 - x^\mu \partial_i T_\mu^i + x^\mu \partial_\nu T_\mu^\nu) \\ &= \int d^d x (T_0^0 + \partial_i (x^\mu) T_\mu^i + x^\mu \partial_\nu T_\mu^\nu) = \int d^d x (T_\mu^\mu + x^\mu \partial_\nu T_\mu^\nu) \\ &\equiv \int d^d x (\Theta + x^\mu \partial_\nu T_\mu^\nu), \end{aligned} \quad (5.10)$$

where we have assumed that the scale current vanishes at spatial infinity, or that the spatial directions have no boundary. The spacetime dependence of the couplings u implies that the stress tensor is *not* conserved; thus, the last equation will not in general vanish.

Now, we set¹⁶ $\Theta = -\beta^a \mathcal{O}_a \epsilon^{\Delta_a - d}$, $\partial_a \equiv \frac{\partial}{\partial u^a}$, and $\partial_a \mathcal{O}_k = B_{ak}^b \mathcal{O}_b$, to find that:

$$\begin{aligned} & \int d^d x \langle \psi_+ | T(\Theta(\mathbf{x}) \mathcal{O}_1 \dots \mathcal{O}_n) | \psi_- \rangle = -i \beta^a \partial_a \langle \psi | T(\mathcal{O}_1 \dots \mathcal{O}_n) | \psi_- \rangle \\ &+ i \sum_k \beta^b B_{kb}^a \langle \psi_+ | T(\mathcal{O}_1 \dots \mathcal{O}_k \dots \mathcal{O}_n) | \psi_- \rangle - \beta^b \langle \psi_+ | \hat{B}_b T(\mathcal{O}_1 \dots \mathcal{O}_n) | \psi_- \rangle \\ &+ \langle \psi_+ | T(\mathcal{O}_1 \dots \mathcal{O}_n) \hat{B}_b | \psi_- \rangle \beta^b, \end{aligned} \quad (5.11)$$

¹⁶The factor of the cutoff is needed for Θ to have dimension d .

where $i\partial_b|\psi\rangle = \hat{B}_b|\psi\rangle$, $-i\partial_b\langle\psi| = \langle\psi|\hat{B}$. With the factor of i , \hat{B} is Hermitian.

Finally, if we define:

$$\begin{aligned}\hat{D}_a\mathcal{O}_a &= \Gamma_a^k\mathcal{O}_k \\ \gamma_k^a &= \Gamma_k^a - \beta^b B_{kb}^a \\ \hat{K}(t) &= \left[Q(t) + \beta^b \hat{B}_b\right]\end{aligned}\tag{5.12}$$

then the Callan-Symanzik equation is:

$$\begin{aligned}&\sum_k \mathbf{x}_k \cdot \partial_{\mathbf{x}_k} \langle\psi_+|T(\mathcal{O}_1 \dots \mathcal{O}_n)|\psi_-\rangle + \sum_k \gamma_k^a \langle\psi_+|T(\mathcal{O}_1 \dots \mathcal{O}_a(\mathbf{x}_k) \dots \mathcal{O}_n)|\psi_-\rangle \\ &+ \beta^a \partial_a \langle\psi_+|T(\mathcal{O}_1 \dots \mathcal{O}_n)|\psi_-\rangle + \int d^d x \langle\psi_+|T(x^\mu \partial_\nu T_\mu^\nu(\mathbf{x}) \mathcal{O}_1 \dots \mathcal{O}_n)|\psi_-\rangle \\ &= i \langle\psi_+|\hat{K}(t_+)T(\mathcal{O}_1 \dots \mathcal{O}_n)|\psi_-\rangle - i \langle\psi_+|T(\mathcal{O}_1 \dots \mathcal{O}_n)\hat{K}(t_-)|\psi_-\rangle.\end{aligned}\tag{5.13}$$

The last term on the second line is the local modification of the Callan-Symanzik equation due to the spacetime-dependent couplings. The two boundary terms on the final line are the modification of the Callan-Symanzik equation for general matrix elements of time-ordered products of operators.¹⁷

To match (4.31) more precisely, let us consider the contribution of the scalar operators in (5.4) to $x^\mu \partial_\nu T_\mu^\nu$. The arguments of Noether's theorem applied to the perturbed CFT leads to the equation

$$x^\mu \partial_\nu T_\mu^\nu = - \sum_a \epsilon^{\Delta-d} (\mathbf{x} \cdot \partial_{\mathbf{x}} u^a(\mathbf{x})) \mathcal{O}_a(\mathbf{x}).\tag{5.14}$$

Now, if we define

$$\tilde{\beta}^a = \beta^a - \mathbf{x} \cdot \partial_{\mathbf{x}} u^a(\mathbf{x}),\tag{5.15}$$

for the scalar operators, and replace β with $\tilde{\beta}$ in (5.12), then we can write the Callan-Symanzik equation in the form:

$$\begin{aligned}&\sum_k \mathbf{x}_k \cdot \partial_{\mathbf{x}_k} \langle\psi_+|T(\mathcal{O}_1 \dots \mathcal{O}_n)|\psi_-\rangle + \sum_k \gamma_k^a \langle\psi_+|T(\mathcal{O}_1 \dots \mathcal{O}_a(\mathbf{x}_k) \dots \mathcal{O}_n)|\psi_-\rangle \\ &+ \tilde{\beta}^a \partial_a \langle\psi_+|T(\mathcal{O}_1 \dots \mathcal{O}_n)|\psi_-\rangle + \int d^d x \langle\psi_+|T(x^\mu \partial_\nu \delta T_\mu^\nu(\mathbf{x}) \mathcal{O}_1 \dots \mathcal{O}_n)|\psi_-\rangle \\ &= i \langle\psi_+|\hat{K}(t_+)T(\mathcal{O}_1 \dots \mathcal{O}_n)|\psi_-\rangle - i \langle\psi_+|T(\mathcal{O}_1 \dots \mathcal{O}_n)\hat{K}(t_-)|\psi_-\rangle,\end{aligned}\tag{5.16}$$

where δT is the contribution of the non-scalar operators. Note that the shift from β to $\tilde{\beta}$ is precisely what we find in the second line of (4.31).

5.3 Lorentzian HJ/CS correspondence

Next, we must understand how the Hamilton-Jacobi equations are related to the Callan-Symanzik equations. To do so, we will carry over the strategy of §4.3 for solving the

¹⁷If we define the states as integrals of local operators acting on the vacuum, then these last two terms can be derived from the Callan-Symanzik equation for vacuum correlators.

Hamilton-Jacobi equations. The major difference is that we must include the dependence on the states.

As in §4, the SUGRA action can be written in a derivative expansion. The regularized generating function Γ is then obtained by the subtraction of some local counterterms $S^{(0)}$ and $S^{(2)}$: $\Gamma = S - (S^{(0)} + S^{(2)})$. Apart from the signature difference, the functional form of the bulk Lagrangian is the same as the Euclidean one. For general states there may be additional boundary terms ψ_{\pm} at Σ_{\pm} in the supergravity action [21]. However, if we work with eigenstates of the supergravity field operators at Σ_{\pm} , these boundary terms will not be present. Lorentzian signature adds no additional ambiguities to the solutions of the first two equations in (4.12). Therefore, the functional form of the local counterterms $S^{(0)}, S^{(2)}$ (4.10) are the same as in the Euclidean case. Furthermore, the functional differential equation in the last line of (4.10) is the same. All of the dependence on the constant of motion \mathbf{a} (and therefore on the state) is contained in the choice of solutions Γ to (4.10). The field theory dual of this statement is that near the UV fixed point, the local beta functions are independent of the state, while the correlation functions depend strongly on the states, especially when the operators are widely separated.

To relate the modified Callan-Symanzik equation (5.16) to the HJ equation, we should adapt the derivation of (4.31) to the case of Lorentzian signature. Let us write the boundary coordinates as $\mathbf{x} = (t, \vec{x})$, where t is the time direction. The correlation function written as a function of all of the dimensionful parameters of the problem is:

$$\begin{aligned} \Gamma_n &\equiv \langle \psi_+(t_+) | T(\mathcal{O}_1(\mathbf{x}_1) \dots \mathcal{O}_n(\mathbf{x}_n)) | \psi_-(t_-) \rangle_c \\ &= \Gamma_n(\mathbf{x}_i, \rho(\mathbf{x}), \tilde{g}_{\mu\nu}(\mathbf{x}), \phi(\mathbf{x}), \psi_+, t_+, \psi_-, t_-) \end{aligned} \quad (5.17)$$

Using the same strategy used in §4.4, we find that:

$$\begin{aligned} &\sum_{k=1}^n \mathbf{x}_k \cdot \frac{\partial}{\partial \mathbf{x}_k} \langle \psi_+ | O_{a_1}(\mathbf{x}_1) \dots O_{a_n}(\mathbf{x}_n) | \psi_- \rangle_c \\ &+ \int d^d y \left[\sum_b \left(-\mathbf{y} \cdot \frac{\partial}{\partial \mathbf{y}} \phi^b(\mathbf{y}) + \beta^b(\phi) \right) \frac{\delta}{\delta \phi^b(\mathbf{y})} \right] \langle \psi_+ | O_{a_1}(\mathbf{x}_1) \dots O_{a_n}(\mathbf{x}_n) | \psi_- \rangle_c \\ &- \int d^d y \left[\left(\mathbf{y} \cdot \frac{\partial}{\partial \mathbf{y}} \rho(\mathbf{y}) \right) \frac{\delta}{\delta \rho(\mathbf{y})} + \delta \tilde{g}_{\mu\nu} \frac{\delta}{\delta \tilde{g}_{\mu\nu}(\mathbf{y})} \right] \langle \psi_+ | O_{a_1}(\mathbf{x}_1) \dots O_{a_n}(\mathbf{x}_n) | \psi_- \rangle_c \\ &+ \sum_{n=1}^k \left(\delta_{a_k}^{b_k} \Delta_{a_k} - \tilde{\gamma}_{a_k}^{b_k}(\phi) \right) \langle \psi_+ | O_{a_1}(\mathbf{x}_1) \dots O_{b_k}(\mathbf{x}_k) \dots O_{a_n}(\mathbf{x}_n) | \psi_- \rangle_c \\ &= \left[\int d^{d-1} \vec{y} dz \left\{ \left[\left(\vec{y} \cdot \frac{\partial}{\partial \vec{y}} + z \partial_z \right) \psi_- \right] \frac{\delta}{\delta \psi_-} + \left[\left(\vec{y} \cdot \frac{\partial}{\partial \vec{y}} + z \partial_z \right) \psi_+ \right] \frac{\delta}{\delta \psi_+} \right\} \right. \\ &\left. - t_- \frac{\partial}{\partial t_-} - t_+ \frac{\partial}{\partial t_+} \right] \langle \psi_+ | O_{a_1}(\mathbf{x}_1) \dots O_{a_n}(\mathbf{x}_n) | \psi_- \rangle_c . \end{aligned} \quad (5.18)$$

The first four lines of this equation are as before. We claim that the final two lines in (5.18) should be precisely dual to the large-N limit of the final line in (5.13), (5.16). To see this, let us consider the supergravity dual of $Q(t_-) | \psi_- \rangle$. In the bulk, the state is specified by a set of functions $\phi_-^a(\vec{x}, z)$ where z is the radial coordinate along Σ_- . These functions are

the eigenstates of the field operators. One may decompose these field operators locally into modes labelled by the \vec{x} -momentum \mathbf{k} , and the *frequency* ω . In the bulk, the equations of motion relate the \vec{k} , ω -dependence to the \vec{x} , z dependence. This map has been made explicit in the large- N limit of unperturbed AdS spacetimes in [7, 32]. In general all we need is that such a map exists, and that the creation operators of the bulk fields are a function of the Fourier modes of the boundary operators.

Now on the boundary, $\mathcal{P}_x = x^i \hat{P}_i \equiv \int d^{d-1} \vec{x} x^i T_i^0$ generates rescalings of the coordinates \vec{x} . It also rescales the Hamiltonian: the Hamiltonian is an operator with mass dimension 1, implying that $[Q(t_-), H(t_-)] = iH(t_-)$. Since H commutes with itself, this implies that $[\mathcal{P}_x, H] = iH$. Now, $|\psi_- \rangle$ is an eigenstate of some function of the Fourier modes of the local operators of the boundary theory. The frequency of these operators is defined by the equation

$$[H, \mathcal{O}_\omega] = i\omega \mathcal{O}_\omega, \quad (5.19)$$

and the momenta by their x -dependence. The result is that \mathcal{P}_x rescales all of the four-momenta of boundary operators. The map to the bulk implies that $\mathcal{P}_x |\psi_- \rangle$ acts via the penultimate line in (5.18).

Finally, we can use (3.1) to note that

$$\langle \psi_+(t_+) | T \exp \left(\frac{i}{\hbar} \int_{t=t_-}^{t=t_+} d^d x \sum_a \lambda^a(\mathbf{x}) \mathcal{O}_a(\mathbf{x}) \right) t_- H(t_-) | \psi_-(t_-) \rangle = t_- d_{t_-} Z = t_- \partial_{t_-} Z. \quad (5.20)$$

Therefore, if the duality holds, the explicit time derivatives in the last line of (5.18) should map to the contributions from $t_\pm H(t_\pm)$ in the final line of (5.13).

To see how the bulk and boundary rescalings of the states map to each other, consider the example of the large- N limit of the unperturbed CFT, in the approximation that we can ignore bulk interactions, and as the cutoff is taken to zero. Let the state be a coherent classical state in which a single local, single-trace operator \mathcal{O} has a macroscopic expectation value:

$$\langle \phi_+ | \mathcal{O}(\mathbf{x}) | \phi_- \rangle = \tilde{\phi}(\mathbf{x}) = \Gamma_1(\mathbf{x}), \quad (5.21)$$

In the unperturbed theory, the spacetime equations of motion imply that $\tilde{\phi}(x)$ is a linear functional of ϕ_+ , ϕ_- , and has dimension Δ_+ . This means that we can write:

$$\tilde{\phi}(\mathbf{x}) = \sum_{\epsilon=\pm} \int_{t_\epsilon} d^{d-1} \vec{y} dz F_\epsilon(\mathbf{x}; \vec{y}, z, t_+, t_-) \phi_\epsilon(\vec{y}, z) \quad (5.22)$$

where G is a function of dimension $\Delta + d$. Therefore, dimensional analysis implies that the final line in (5.18) acting on Γ_1 is:

$$\left[\int d^{d-1} \vec{y} dz \left\{ \left[\left(\vec{y} \cdot \frac{\partial}{\partial \vec{y}} + z \partial_z \right) \phi_- \right] \frac{\delta}{\delta \phi_-} + \left[\left(\vec{y} \cdot \frac{\partial}{\partial \vec{y}} + z \partial_z \right) \phi_+ \right] \frac{\delta}{\delta \phi_+} \right\} - t_- \frac{\partial}{\partial t_-} - t_+ \frac{\partial}{\partial t_+} \right] \tilde{\phi}(\mathbf{x}; \phi_+, t_+, \phi_-, t_-) = \left(\mathbf{x} \cdot \frac{\vec{\partial}}{\partial \mathbf{x}} + \Delta \right) \tilde{\phi}. \quad (5.23)$$

The required that we rescaled both the arguments of ϕ_\pm as well as t_\pm .

Now, let us compare this to the right hand side of (5.13). We take the basis of operators such that $\hat{B}_a|\phi_{\pm}\rangle = 0$. Conformal invariance implies that $Q(t)$ is constant in time, so that we can write:

$$\begin{aligned} & i\langle\phi_+|Q(t_+) \mathcal{O}(\mathbf{x})|\phi_-\rangle - i\langle\phi_+|\mathcal{O}(t,x)Q(t_-)|\phi_-\rangle \\ &= i\langle\phi_+|[Q, \mathcal{O}(\mathbf{x})]|\phi_-\rangle = \langle\phi_+|\left(\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{x}} + \Delta\right)\mathcal{O}(\mathbf{x})|\phi_-\rangle = \left(\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{x}} + \Delta\right)\tilde{\phi}. \end{aligned} \quad (5.24)$$

This is precisely the variation of $\tilde{\phi}$ that we find in (5.18), (5.23). Note further that (5.18) has only been derived for classical states in the large- N limit. For such states in the field theory, the action of Q on the states will be specified by the scale transformation of the one-point functions.

We have solved the main problem raised in the introduction. Let us summarize the basic point. The Hamilton-Jacobi formalism involves two sets of equations. One set of equations is Hamilton's equations for the field variables $\phi(x, z)$, (roughly) $z\partial_z\phi = \frac{\delta S}{\delta\phi}$. Studies of holographic RG in Euclidean signature indicate that these equations are mapped into the standard first-order RG flow equations $\Lambda\partial_\Lambda\lambda^a = \beta^a$, at least near the timelike boundary of the 5d spacetime. Given the HJ functional S , the solutions are completely specified by the values of ϕ at some UV scale z . For $z \rightarrow 0$, this data is dual to the UV couplings λ^a .

However, the spacetime equations of motion are second order (at large N and low energies.) In Lorentzian signature their solutions are only uniquely specified after one specifies additional data, dual to the presence of normalizable modes. The point is that to solve the spacetime equations of motion one solves the Hamilton-Jacobi equation for S , and *then* solves Hamilton's equations for ϕ . S depends on additional constants of motion \mathbf{a} . In the discussion above we have found that \mathbf{a} labels the classical states of the theory.

We have shown that the Hamilton-Jacobi and Callan-Symanzik equations for correlation functions are identical (in the large- N limit) even in the presence of classical states. Therefore the constants of motion \mathbf{a} in the Hamilton-Jacobi formulation of supergravity are precisely dual to data required in the field theory, in order to specify the scaling behavior of correlation functions. In other words, the field theory contains all the structure of a theory governed by second order equations of motion in the bulk. The scaling behavior of the theory is determined by both the RG flow equations and the Callan-Symanzik equations.

One caveat is that while the natural RG flow equations in field theory specify the variation of the couplings with scale, the associated Hamilton's equations of the supergravity dual specify the flow of ϕ with z . These two statements are only precisely dual as $z \rightarrow 0$. We will discuss the case of finite z in §5.5.

5.4 Perturbed CFTs with an IR cutoff

We have studied solutions to the Hamilton-Jacobi equations with constants of motion that specify the (classical) state of the system. There are other solutions. We will discuss one class for which the constants of motion are dual to one-point functions of operators

specified at some infrared scale. These are not the same: as discussed in [7], these one-point functions depend on both the couplings, the state, and the scale.

We consider single free scalar field excitations at energies much smaller than the Planck scale. For such low energy excitations backreaction is neglectable and one can treat the scalar field as it was propagating in a fixed AdS background. In this limit Hamilton's principal function $S[(g_{\mu\nu}(\mathbf{x}), \phi(\mathbf{x})), \mathbf{a}]$ reduces to a function of $(z, \phi(\mathbf{x}), \mathbf{a})$, where the metric dependence is replaced by dependence on z . The corresponding HJ equation reduces to a form similar to (2.7), where t is replaced by z .

One solution to the HJ equation, analogous to (2.6), is

$$S[\phi(\mathbf{x}), \mathbf{a}, z] = \int_z^{z_0} d\tilde{z} \int d^d x \mathcal{L}(\phi(\mathbf{x}, \tilde{z})), \quad (5.25)$$

where $z_0 > z$, evaluated on a solution to the equation of motion with boundary conditions $\phi(\mathbf{x}, z) = \phi(\mathbf{x})$, $\mathbf{a} \equiv \phi(\mathbf{x}, z_0) = \phi_0(\mathbf{x})$. For z_0 deep enough in the bulk, ϕ is dominated by the term scaling as $z^{\Delta+}$. Therefore, $z_0^{-\Delta+} \phi_0$ can be interpreted as the one-point function of the operator at scale z_0 . Note, however, that with this choice for the constants of motion \mathbf{a} held fixed, the nonlocal part of S in (5.25) is no longer the generating function of correlation functions. In order to keep the expectation values ϕ_0 fixed as one varies $\phi(\mathbf{x})$, one must vary the couplings *and* the state.

Our simplified Lagrangian is

$$\mathcal{L}(\phi(\mathbf{x}, \tilde{z})) = -\frac{1}{2} \tilde{z}^{1-d} \left[(\partial_{\tilde{z}} \phi)^2 + \eta^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{m^2}{z^2} \phi^2 \right]. \quad (5.26)$$

Integrating the kinetic term by parts and using the equation of motion we find that (5.25) can be written as two boundary terms:

$$S[\phi(\mathbf{x}), \phi_0(\mathbf{x})] = -\frac{1}{2} \int d^d x \tilde{z}^{1-d} \phi \partial_{\tilde{z}} \phi \Big|_z^{z_0} = \frac{1}{2} \int d^d x \left[z^{1-d} \phi \partial_z \phi - z_0^{1-d} \phi_0 \partial_{z_0} \phi_0 \right]. \quad (5.27)$$

For simplicity, let us consider a scalar field that almost saturates the Breitenlohner-Freedman bound $R_{\text{AdS}}^2 m^2 \geq -4$ [28], or equivalently $0 < \nu \ll 1$. The solutions to the equations of motion will have the following leading behavior at small $z \leq z_0 \ll R_{\text{AdS}}$

$$\phi(\mathbf{x}, z) = \alpha(\mathbf{x}) z^{\Delta-} (1 + O(z^2) + \dots) + \beta(\mathbf{x}) z^{\Delta+} (1 + O(z^2) + \dots). \quad (5.28)$$

In this case both independent solutions are normalizable. One may choose either of α, β to be dual to the coupling, with Δ_{\mp} the corresponding operator dimension [48]. These are related by a Legendre transformation [18, 39]. We will consider the case that the operator dimension is Δ_+ , and α is dual to the coupling in the field theory. The discussion should then connect smoothly to one for operators of higher dimension, for which the term proportional to α is non-normalizable.

To write (5.27) as a solution to the Hamilton-Jacobi equations, we must find $\partial_z \phi, \partial_{z_0} \phi_0$ as a function of ϕ, ϕ_0 :

$$z \partial_z \phi \simeq \left[\frac{\Delta_- z^{-2\nu}}{z^{-2\nu} - z_0^{-2\nu}} + \frac{\Delta_+ z^{2\nu}}{z^{2\nu} - z_0^{2\nu}} \right] \phi - \left[\frac{\Delta_- z^{\Delta_-} z_0^{-\Delta_+}}{z^{-2\nu} - z_0^{-2\nu}} + \frac{\Delta_+ z^{\Delta_+} z_0^{-\Delta_-}}{z^{2\nu} - z_0^{2\nu}} \right] \phi_0 z_0 \partial_{z_0} \phi_0$$

$$\simeq - \left[\frac{\Delta_- z_0^{-2\nu}}{z^{-2\nu} - z_0^{-2\nu}} + \frac{\Delta_+ z_0^{2\nu}}{z^{2\nu} - z_0^{2\nu}} \right] \phi_0 + \left[\frac{\Delta_- z^{-\Delta_+} z_0^{\Delta_-}}{z^{-2\nu} - z_0^{-2\nu}} + \frac{\Delta_+ z^{-\Delta_-} z_0^{\Delta_+}}{z^{2\nu} - z_0^{2\nu}} \right] \phi. \quad (5.29)$$

The classical action (5.27) can be written as:

$$S \simeq \int d^d x \left\{ \frac{1}{2} \left[\frac{\Delta_- z^{-d-2\nu}}{z^{-2\nu} - z_0^{-2\nu}} + \frac{\Delta_+ z^{-d+2\nu}}{z^{2\nu} - z_0^{2\nu}} \right] \phi^2 - \left[\frac{\Delta_- (z z_0)^{-\Delta_+}}{z^{-2\nu} - z_0^{-2\nu}} + \frac{\Delta_+ (z z_0)^{-\Delta_-}}{z^{2\nu} - z_0^{2\nu}} \right] \phi \phi_0 + \frac{1}{2} \left[\frac{\Delta_- z_0^{-d-2\nu}}{z^{-2\nu} - z_0^{-2\nu}} + \frac{\Delta_+ z_0^{-d+2\nu}}{z^{2\nu} - z_0^{2\nu}} \right] \phi_0^2 \right\}. \quad (5.30)$$

As $z \rightarrow 0$ the dimensionless UV coupling becomes $u = \alpha z^{\Delta_-}$. The dimensionless one-point function of the dual operator at scale z_0 is $\tilde{u} = \beta z_0^{\Delta_+}$. We take $u, \tilde{u} \sim 1$: the coupling is specified at a UV scale, and the one-point function at some IR scale. With u, \tilde{u} so specified, β will dominate over α at $z_0 \gg z$, as $\beta z_0^{\Delta_+} \gg \alpha z_0^{\Delta_-}$ when $\tilde{u} \gg u \left(\frac{z}{z_0}\right)^{\Delta_-}$. If we take the limits $z_0 \ll R_{\text{AdS}}$ and $z/z_0 \ll 1$, we will find that the interpretation of flow in z as RG flow is particularly clean. In particular, the action simplifies:

$$S \sim \int d^d x \left[\frac{1}{2} \Delta_- z^{-d} \phi^2 + 2\nu z^{-\Delta_-} z_0^{-\Delta_+} \phi \phi_0 - \frac{1}{2} \Delta_+ z_0^{-d} \phi_0^2 \right]. \quad (5.31)$$

This satisfies the Hamilton-Jacobi equation in the limit specified above, by construction. Next, Hamilton's equations for ϕ are:¹⁸

$$-z^d \pi_\phi = z \partial_z \phi = z^d \frac{\delta S}{\delta \phi} \sim \Delta_- \phi + 2\nu z^{\Delta_+} z_0^{-\Delta_+} \phi_0. \quad (5.32)$$

The term $\Delta_- \phi$ is just the beta function to linear order. The second, subleading term, controls the one-point function for the dual operator. The general solution to (5.32) is:

$$\phi(\mathbf{x}, z) \sim z^{\Delta_-} \lambda(\mathbf{x}) + z^{\Delta_+} z_0^{-\Delta_+} \phi_0(\mathbf{x}), \quad (5.33)$$

where $\lambda(\mathbf{x})$ corresponds to the coupling.

In this limit, (5.25) has the form of the solutions discussed in §3. Here $\frac{1}{2} \Delta_- \phi^2$ matches the "local" contribution $S^{(0)}$. The other two terms in (5.31) belong to Γ . Note that

$$\partial_\phi \Gamma|_{\phi_0 \text{ fixed}} = 2\nu z^{-\Delta_-} z_0^{-\Delta_+} \phi_0 = \langle \mathcal{O} \rangle. \quad (5.34)$$

However, $\partial_\phi^2 \Gamma|_{\phi_0 \text{ fixed}} = 0$. Because we have kept ϕ_0 rather than $\tilde{\phi}$ in our variations, variations of Γ with respect to ϕ are not the correlation functions of the theory.

The action (5.30) has a symmetry under exchanging (z_0, ϕ_0, Δ_+) with (z, ϕ, Δ_-) and flipping the overall sign. Thus, taking the opposite limit $z_0/z \rightarrow 0$ with $z \ll R_{\text{AdS}}$, will result in exchanging the roles of ϕ_0 and λ . This limit matches the discussion of §3 if we choose the second root $\vartheta_a = \Delta_{a,+}$ in (4.19).

We should note that this solution to the Hamilton-Jacobi equation could just as easily have been found in Euclidean space. The fields will be nonsingular in the region between

¹⁸Note that since z is the lower bound of the integral (5.25), $\pi_\phi = -\frac{\delta S}{\delta \phi}$.

z, z_0 . Only when the IR cutoff is removed will we be forced to choose a particular value for the normalizable mode. This statement has an candidate analog in the dual field theory. Conformal perturbation theory for relevant perturbations is plagued by infrared divergences: the proper treatment of these divergences requires adjusting the one-point functions of the theory (*c.f.* [49, 50]-[51].)

5.5 Extension of the holographic RG equations into the IR

As with other discussions of the holographic RG formalism, ours has taken place deep in the UV region ($z \rightarrow 0$) of the theory. There are a number of issues with extending the equations into the IR, some of which become even more difficult in the Lorentzian description. Many of these have been mentioned elsewhere, but we wish to collect the problems here and expand on them.

1. The identification of $\phi(\mathbf{x}, z)$ as the dimensionless coupling at some scale $l(r)$ was based on the asymptotic behavior of ϕ as $z \rightarrow 0$ in (3.5). For finite z , the relation between ϕ and the coupling is more complicated. (See also [19, 20] for a discussion of this issue.) First, eq. (3.15) shows that ϕ is in general a sum of contributions from the couplings and contributions from the state. Since the procedure of integrating out modes will depend on the properties of the state (near the cutoff), one might imagine that as one lowers the cutoff (identified with z), ϕ can be identified with the renormalized coupling in some scheme. However, it is unclear to us that such a scheme exists and is useful in the dual field theory.

Even without such a scheme, some sort of relationship between the Hamilton-Jacobi and Callan-Symanzik equations should hold. However, the simple local, linear relation used in (4.25) will no longer hold. More generally, $\rho^{-\Delta} \frac{\delta}{\delta\phi(\mathbf{x}, z)}$ in (4.25) should be replaced with

$$\int d^d y \frac{\delta\phi(\mathbf{y}, z)}{\delta\tilde{\lambda}_z(\mathbf{x})} \frac{\delta}{\delta\phi(\mathbf{y}, z)}, \tag{5.35}$$

where $\tilde{\lambda}_z$ is the dimensionless coupling at scale z , and the derivative $\delta_{\tilde{\lambda}_z} \phi$ is taken with the state fixed.

Nonetheless let us continue to discuss the Hamilton-Jacobi equations. The scaling of the fields identified by [15, 16] becomes complicated at finite z , as the authors of those references indeed point out. This leaves less of a reason to solve the Hamilton-Jacobi equations by breaking them up as in (4.12). Nonetheless, the derivative expansion in the bulk remains valid, and the first two lines of (4.12) still have the same solution as before. However, there were additional terms in the Hamilton-Jacobi equation that were dropped in the small- z approximation, that we can no longer drop. These modify the third equation in (4.12). The Hamilton-Jacobi equations become:

$$\begin{aligned} \{S^{(0)}, S^{(0)}\} &= \mathcal{L}^{(0)} \\ 2\{S^{(0)}, S^{(2)}\} &= \mathcal{L}^{(2)} \\ 2\{S^{(0)} + S^{(2)} + \Gamma, \Gamma\} - \{\Gamma, \Gamma\} &= -\{S^{(2)}, S^{(2)}\} . \end{aligned} \tag{5.36}$$

where $\Gamma \equiv S - S^{(0)} - S^{(2)}$. We will assume that Γ is the generating function of correlation functions in the cutoff theory. The first term in the third equation can be rewritten via the full set of Hamilton's equations

$$\begin{aligned} \frac{\partial \phi^a(\mathbf{x}, r)}{\partial r} &= \frac{G^{ab}(\phi)}{\sqrt{g}} \frac{\delta}{\delta \phi^a(\mathbf{x}, r)} \left[S^{(0)} + S^{(2)} + \Gamma \right] \\ \frac{\partial g_{\mu\nu}(\mathbf{x}, r)}{\partial r} &= \frac{1}{\sqrt{g}} \left(-2 \frac{\delta}{\delta g^{\mu\nu}(\mathbf{x}, r)} + \frac{2}{3} g_{\mu\nu} g^{\lambda\rho} \frac{\delta}{\delta g^{\lambda\rho}(\mathbf{x}, r)} \right) \left[S^{(0)} + S^{(2)} + \Gamma \right], \end{aligned} \quad (5.37)$$

such that

$$\begin{aligned} \left(-\frac{d}{dr} + \frac{\partial}{\partial r} \right) \Gamma &\equiv \int d^d x \left[\frac{\partial g_{\mu\nu}}{\partial r} g^{\mu\rho} g^{\nu\sigma} \frac{\delta}{\delta g^{\rho\sigma}(\mathbf{x}, r)} - \frac{\partial \phi^a}{\partial r} \frac{\delta}{\delta \phi^a(\mathbf{x}, r)} \right] \Gamma \\ &= - \int d^d x \left\{ S^{(2)}, S^{(2)} \right\} + \int d^d \mathbf{x} \{ \Gamma, \Gamma \}. \end{aligned} \quad (5.38)$$

Eq. (4.4) states that $\partial_r S = 0$. Since we can see explicitly that $\partial_r(S^{(0)} + S^{(2)}) = 0$, this implies that $\partial_r \Gamma = 0$ as well, leaving us with the tantalizing equation:

$$d_r \Gamma + \int d^d x \{ \Gamma, \Gamma \} = \int d^d x \{ S^{(2)}, S^{(2)} \}. \quad (5.39)$$

We leave the field-theoretic interpretation of this equation for future work. Note that without the $\{ \Gamma, \Gamma \}$ term, this looks like an integrated form of Osborn's version of the Callan-Symanzik equations [41].

There are two further problems with relating (5.39) to the field theoretic Callan-Symanzik equations.

1. As discussed in point (1) above, the relation between ϕ and the couplings as typically defined may be complicated, and requires information about the quantum state. Therefore, there is a lot of work to relate $\partial_r \phi \delta_\phi \Gamma$ to $\beta \partial_\lambda \Gamma$. Note that for spacetime dependent couplings, one *does* expect nonlocal contributions to the beta function (as mentioned in [41]), so some piece of the contribution to $\partial_r \phi$, $\partial_r g$ from $\partial_\phi \Gamma$ and $\partial_g \Gamma$ may appear in the field-theoretic beta functions.
2. We have considered deformations in the UV by single-trace operators only. However, multiple-trace operators will generically be induced under the RG flow [24]. Nonetheless, consider the (infinite-dimensional) surface in the space of couplings which is swept out by RG trajectories which are purely single-trace in the ultraviolet. For our Hamilton-Jacobi equations to successfully capture the large-N RG equations, we are assuming that in our scheme, $\phi^a(\mathbf{x}, z)$ are good coordinates on this surface.
3. The role of the term $\{ \Gamma, \Gamma \}$ on the left hand side of (5.39) is not understood.

We leave these issues for future work.

5.6 Reversibility of holographic RG

The work of Susskind and Witten [2] suggests that "cutting off" the asymptotic region $z < \epsilon$ of AdS space is dual to a spatial cutoff in the dual field theory. However, Wilsonian renormalization, achieved by tracing out degrees of freedom at scales larger than the cutoff, is not a reversible process, while the second order supergravity equations can be integrated either out towards the boundary or in towards the interior. If S_{reg} in (4.1) or (5.1) is meant to describe the quantum field theory cut off at distance scale z_{UV} , why then can we integrate the equations of motion out to the boundary?

In the classical limit of the spacetime theory, the answer is that the cutoff in S_{reg} merely smooths out the short-distance singularities of the theory, and does not set them to zero. For example, we can see that the two-point function is smooth and generally nonvanishing as the separation vanishes.¹⁹ In our discussion until this point, no limitation has been placed on the sensitivity of our measurements, so that we can specify information about the theory at all scales, even in the presence of a cutoff. If this information includes all possible irrelevant operators (dual to massive fields in the bulk spacetime), then we can follow the theory into the UV without any obstruction.

In practice, detectors sensitive to gauge theory observables will have limited accuracy. The detectors could have finite spatial resolution ℓ , or they could have finite sensitivity to the amplitude of the fluctuations. More generally both limitations will be in effect. In the former case, one would naturally perform experiments with z_{UV} set equal to ℓ ; the finite resolution makes it impossible to follow the theory into the UV. In the second case, at any given cutoff z_{UV} , one cannot study correlations much below that cutoff, so that one cannot follow the coupling into the UV.

Either way, limits on the accuracy of our detectors are not built into our classical, large- N discussion of the AdS/CFT correspondence; this is why we have seen no hint of irreversibility in our discussion. Of course, one could also study the duals of gauge theories which are explicitly cut off, as in [52, 53].

6. Conclusions

We have resolved the apparent tension between the first-order RG equations of a quantum field theory and the second-order supergravity equations which are supposed to encode the RG flow in the dual asymptotically-AdS spacetime. The essential point is that the RG behavior of the field theory is contained in *two* first-order equations — the Callan-Symanzik equations, and the equations for the evolution of the couplings. The former depends on the choice of quantum state, which is the additional information one needs to specify the most general solution to the bulk, second-order supergravity equations.

A number of puzzles remain. In particular, we would like to better understand the relationship between the bulk fields and the boundary coupling deep in the IR, as discussed in §4.5; and we would like to understand the apparent modification of the Callan-Symanzik equations (including the $\{\Gamma, \Gamma\}$ term) in eq. (5.39).

¹⁹The point that the finite- z cutoff is a complicated "smearing" function has been made, for example, in [14].

Acknowledgments

We would like to thank C. Beasley, M. Headrick, D. Kabat, H. Liu, M.S. Sheikh-Jabbari, and J. Terning for helpful conversations. We would particularly like to thank O. Aharony, J. de Boer, D. Marolf, and H. Schnitzer for comments on the draft, as well as for very helpful discussions and correspondence. We would also like to thank the MIT Center for Theoretical Physics for their hospitality during the completion of this project. A.L. would like to thank D.Z. Freedman and M. Headrick for discussions on related subjects during the incubation of [44]. He would also like to thank the UC Davis cosmology and particle physics groups, the Stanford theory group, and the IPM string theory group for their generous hospitality during parts of this project. The research of A.L. and A.S. is supported in part by NSF grant PHY-0331516, and by DOE Grant No. DE-FG02-92ER40706 via an Outstanding Junior Investigator award.

We would like to dedicate this work to the memories of John Brodie and Andrew Chamblin.

References

- [1] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [[hep-th/9711200](#)].
- [2] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, [hep-th/9805114](#).
- [3] T. Banks and M.B. Green, *Non-perturbative effects in $AdS_5 \times S^5$ string theory and $D = 4$ SUSY Yang-Mills*, *JHEP* **05** (1998) 002 [[hep-th/9804170](#)].
- [4] C.-S. Chu, P.-M. Ho and Y.-Y. Wu, *D-instanton in AdS_5 and instanton in $SYM(4)$* , *Nucl. Phys.* **B 541** (1999) 179 [[hep-th/9806103](#)].
- [5] I.I. Kogan and G. Luzon, *D-instantons on the boundary*, *Nucl. Phys.* **B 539** (1999) 121 [[hep-th/9806197](#)].
- [6] M. Bianchi, M.B. Green, S. Kovacs and G. Rossi, *Instantons in supersymmetric Yang-Mills and D-instantons in IIB superstring theory*, *JHEP* **08** (1998) 013 [[hep-th/9807033](#)].
- [7] V. Balasubramanian, P. Kraus, A.E. Lawrence and S.P. Trivedi, *Holographic probes of anti-de Sitter space-times*, *Phys. Rev.* **D 59** (1999) 104021 [[hep-th/9808017](#)].
- [8] A.W. Peet and J. Polchinski, *UV/IR relations in AdS dynamics*, *Phys. Rev.* **D 59** (1999) 065011 [[hep-th/9809022](#)].
- [9] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large- N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C 22** (2001) 379 [[hep-th/9803001](#)].
- [10] J.M. Maldacena, *Wilson loops in large- N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [[hep-th/9803002](#)].
- [11] E.T. Akhmedov, *A remark on the AdS/CFT correspondence and the renormalization group flow*, *Phys. Lett.* **B 442** (1998) 152 [[hep-th/9806217](#)].
- [12] E. Alvarez and C. Gomez, *Geometric holography, the renormalization group and the c-theorem*, *Nucl. Phys.* **B 541** (1999) 441 [[hep-th/9807226](#)].

- [13] M. Porrati and A. Starinets, *Rg fixed points in supergravity duals of 4D field theory and asymptotically AdS spaces*, *Phys. Lett.* **B 454** (1999) 77 [[hep-th/9903085](#)].
- [14] V. Balasubramanian and P. Kraus, *Spacetime and the holographic renormalization group*, *Phys. Rev. Lett.* **83** (1999) 3605 [[hep-th/9903190](#)].
- [15] J. de Boer, E.P. Verlinde and H.L. Verlinde, *On the holographic renormalization group*, *JHEP* **08** (2000) 003 [[hep-th/9912012](#)].
- [16] J. de Boer, *The holographic renormalization group*, *Fortschr. Phys.* **49** (2001) 339 [[hep-th/0101026](#)].
- [17] V. Balasubramanian, P. Kraus and A.E. Lawrence, *Bulk vs. boundary dynamics in anti-de Sitter spacetime*, *Phys. Rev.* **D 59** (1999) 046003 [[hep-th/9805171](#)].
- [18] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [[hep-th/9905104](#)].
- [19] J. Kalkkinen, D. Martelli and W. Muck, *Holographic renormalisation and anomalies*, *JHEP* **04** (2001) 036 [[hep-th/0103111](#)].
- [20] M. Berg and H. Samtleben, *Holographic correlators in a flow to a fixed point*, *JHEP* **12** (2002) 070 [[hep-th/0209191](#)].
- [21] D. Marolf, *States and boundary terms: subtleties of lorentzian AdS/CFT*, *JHEP* **05** (2005) 042 [[hep-th/0412032](#)].
- [22] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [[hep-th/9802109](#)].
- [23] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)].
- [24] O. Aharony, M. Berkooz and E. Silverstein, *Multiple-trace operators and non-local string theories*, *JHEP* **08** (2001) 006 [[hep-th/0105309](#)].
- [25] E. Witten, *Multi-trace operators, boundary conditions and AdS/CFT correspondence*, [hep-th/0112258](#).
- [26] M. Berkooz, A. Sever and A. Shomer, *Double-trace deformations, boundary conditions and spacetime singularities*, *JHEP* **05** (2002) 034 [[hep-th/0112264](#)].
- [27] A. Sever and A. Shomer, *A note on multi-trace deformations and AdS/CFT*, *JHEP* **07** (2002) 027 [[hep-th/0203168](#)].
- [28] P. Breitenlohner and D.Z. Freedman, *Positive energy in anti-de Sitter backgrounds and gauged extended supergravity*, *Phys. Lett.* **B 115** (1982) 197.
- [29] P. Breitenlohner and D.Z. Freedman, *Stability in gauged extended supergravity*, *Ann. Phys. (NY)* **144** (1982) 249.
- [30] G.T. Horowitz and H. Ooguri, *Spectrum of large-N gauge theory from supergravity*, *Phys. Rev. Lett.* **80** (1998) 4116 [[hep-th/9802116](#)].
- [31] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [[hep-th/9803131](#)].
- [32] T. Banks, M.R. Douglas, G.T. Horowitz and E.J. Martinec, *AdS dynamics from conformal field theory*, [hep-th/9808016](#).

- [33] L.G. Yaffe, *Large- N limits as classical mechanics*, *Rev. Mod. Phys.* **54** (1982) 407.
- [34] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large- N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [[hep-th/9905111](#)].
- [35] K. Skenderis and M. Taylor, *Kaluza-Klein holography*, *JHEP* **05** (2006) 057 [[hep-th/0603016](#)].
- [36] J. Polchinski, L. Susskind and N. Toumbas, *Negative energy, superluminosity and holography*, *Phys. Rev. D* **60** (1999) 084006 [[hep-th/9903228](#)].
- [37] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT(d)/AdS($d+1$) correspondence*, *Nucl. Phys. B* **546** (1999) 96 [[hep-th/9804058](#)].
- [38] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, [hep-th/0003136](#).
- [39] T. Hartman and L. Rastelli, *Double-trace deformations, mixed boundary conditions and functional determinants in AdS/CFT*, [hep-th/0602106](#).
- [40] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. and Quant. Grav.* **19** (2002) 5849 [[hep-th/0209067](#)].
- [41] H. Osborn, *Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories*, *Nucl. Phys. B* **363** (1991) 486.
- [42] J. Erdmenger, *A field-theoretical interpretation of the holographic renormalization group*, *Phys. Rev. D* **64** (2001) 085012 [[hep-th/0103219](#)].
- [43] J. Cardy, *Scaling and renormalization in statistical physics*, Cambridge, 1996.
- [44] D.Z. Freedman, M. Headrick and A. Lawrence, *On closed string tachyon dynamics*, *Phys. Rev. D* **73** (2006) 066015 [[hep-th/0510126](#)].
- [45] A.B. Zamolodchikov, *Renormalization group and perturbation theory near fixed points in two-dimensional field theory*, *Sov. J. Nucl. Phys.* **46** (1987) 1090.
- [46] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [[hep-th/9806087](#)].
- [47] M. Fukuma, S. Matsuura and T. Sakai, *A note on the Weyl anomaly in the holographic renormalization group*, *Prog. Theor. Phys.* **104** (2000) 1089 [[hep-th/0007062](#)].
- [48] V. Balasubramanian, S.B. Giddings and A.E. Lawrence, *What do CFTs tell us about anti-de Sitter spacetimes?*, *JHEP* **03** (1999) 001 [[hep-th/9902052](#)].
- [49] K.G. Wilson, *Nonlagrangian models of current algebra*, *Phys. Rev.* **179** (1969) 1499.
- [50] R. Jackiw and S. Templeton, *How superrenormalizable interactions cure their infrared divergences*, *Phys. Rev. D* **23** (1981) 2291.
- [51] A.B. Zamolodchikov, *Two point correlation function in scaling Lee-Yang model*, *Nucl. Phys. B* **348** (1991) 619.
- [52] S. Hellerman, *Lattice gauge theories have gravitational duals*, [hep-th/0207226](#).
- [53] N. Evans, T.R. Morris and O.J. Rosten, *Gauge invariant regularization in the AdS/CFT correspondence and ghost D-branes*, *Phys. Lett. B* **635** (2006) 148 [[hep-th/0601114](#)].